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A SPACE OF POINTWISE COUNTABLE TYPE AND PERFECT MAPS

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ABSTRACT. There exists a Lindelöf space, of pointwise countable type, which does not admit a perfect map onto any space in which every point is G_δ.

Introduction. Recall from [A] that a space is of *pointwise countable type* if each point is contained in a compact set of countable character. In [**O**, Question 7.5], Olson asked if there is a paracompact Hausdorff space, of pointwise countable type, which does not admit a perfect map onto a first countable space. In this note, we answer this question affirmatively by exhibiting an example of a regular Lindelöf space, of pointwise countable type, which does not admit a perfect map onto any space in which every point is G_δ. The example is obtained by adding the closed unit interval to the space M constructed by Dowker in [**D**].

The example. Let I be the closed unit interval, and ω_1 the first uncountable ordinal. For ordinals α, β with $\alpha \leqslant \beta \leqslant \omega_1$, $[\alpha, \beta]$ denotes the space $\{\gamma \mid \alpha \leqslant \gamma \leqslant \beta\}$ of ordinals with the order topology. Let Q be the set of all rational numbers in I, and let $\{Q_\alpha \mid \alpha < \omega_1\}$ be a disjoint collection of countable dense subsets in I consisting of irrational numbers. Consider the product space $[0, \omega_1] \times I$ and its subspace

$$X = ([0, \omega_1] \times I) - \bigcup_{\alpha < \omega_1} ([0, \alpha] \times Q_\alpha).$$

Claim 1. *X is a Lindelöf space of pointwise countable type.*

PROOF. Let $\mathfrak{U}$ be an open cover of X. By using compactness of $\{\omega_1\} \times I$, we can find $\alpha_0 < \omega_1$ such that $([\alpha_0, \omega_1] \times I) \cap X$ is covered by finitely many members of $\mathfrak{U}$. Since $([0, \alpha_0] \times I) \cap X$ satisfies the second axiom of countability, it follows that $\mathfrak{U}$ has a countable subcover. For each $(\alpha, p) \in X$, $([0, \omega_1] \times \{p\}) \cap X$ is a compact set of countable character and contains (α, p). $\square$

Let Y be a space in which every point is G_δ, and let $f\colon X \to Y$ be a continuous map such that $f^{-1}(y)$ is compact for each $y \in Y$. It suffices to show that f is not a closed map. For each $q \in Q$, since $f^{-1}f((\omega_1, q))$ is a G_δ-set, there is $\alpha_q < \omega_1$ such that $[\alpha_q, \omega_1] \times \{q\} \subset f^{-1}f((\omega_1, q))$. Let $\beta = \sup\{\alpha_q \mid q \in Q\}$; then $\beta < \omega_1$. Let us set $J = \{\omega_1\} \times I$.

Claim 2. *For each $y \in Y$, $f^{-1}(y) \cap J$ is nowhere dense in J.*

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PROOF. Suppose that $f^{-1}(y) \cap J$ contains an open interval U in J. Then there exist $p \in Q_\beta$ with $(\omega_1, p) \in U$ and a sequence $\{q_n\}_{n\in N}$ in Q, converging to p, such that $(\omega_1, q_n) \in U$ for each $n \in N$. Let $E = \{(\beta, q_n) \mid n \in N\}$. Then $E \subset f^{-1}(y)$ and E is discrete closed in X since $(\beta, p) \notin X$. This contradicts the fact that $f^{-1}(y)$ is compact. □

Claim 3. *f is not a closed map.*

PROOF. Pick $r \in Q_\beta$, and let $y = f((\omega_1, r))$. By Claim 2, we can find a sequence $\{s_n\}_{n\in N}$ in Q such that $|r - s_n| < 1/n$ and

$$(\omega_1, s_n) \notin f^{-1}(y).$$

Let $y_n = f((\omega_1, s_n))$, for each $n \in N$, and $F = \{(\beta, s_n) \mid n \in N\}$. Then, f being continuous, $\{y_n\}$ converges to y. Since $(\beta, r) \notin X$, F is closed in X. But $f(F)$ is not closed in Y, because $f(F) = \{y_n \mid n \in N\}$. Hence the proof is complete. □

Remarks. Olson's question was repeated by Burke in [**B**], and appears also in [**R**]. It is not possible to strengthen our example by making X locally compact. In fact, X would then be mapped perfectly onto a metrizable space (cf. [**F**, Theorem 3]).

REFERENCES

[A] A. V. Arhangel'skiĭ, *Bicompact sets and the topology of spaces*, Trudy Moskov. Mat. Obšč. **13** (1965), 3–55 = Trans. Moscow Math. Soc. **13** (1965), 1–62.

[B] D. K. Burke, *Closed mappings*, Surveys in General Topology (G. Reed, Ed.), Academic Press, New York, 1980, pp. 1–32.

[D] C. H. Dowker, *Local dimension of normal spaces*, Quart. J. Math. Oxford (2) **6** (1955), 101–120.

[F] Z. Frolík, *On the topological product of paracompact spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 747–750.

[O] R. C. Olson, *Bi-quotient maps, countably bi-sequential spaces and related topics*, General Topology Appl. **4** (1974), 1–28.

[R] M. E. Rudin, *Lectures on set theoretic topology*, CBMS Regional Conf. Ser. Math., vol. 23, Amer. Math. Soc., Providence, R.I., 1975.

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