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PARACOMPACTNESS OF PIXLEY-ROY HYPERSPACES. II

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ABSTRACT. In this paper we will study the paracompactness of Pixley-Roy hyperspaces of products. We will prove that if $\mathcal{F}[X]$ is paracompact (resp. hereditarily paracompact), then $\mathcal{F}[X^2]$ is paracompact (resp. hereditarily paracompact). This is the positive answer to T. Przymusiński's problem.

1. Introduction. Throughout this paper we will assume that X is a T_1-space. This paper continues the study of the paracompactness of Pixley-Roy hyperspaces initiated in [8]. Recall that for each space X, the space $\mathcal{F}[X]$, called the *Pixley-Roy hyperspace* over X, defined by C. Pixley and P. Roy in [5], is the set of all nonempty finite subsets of X with the topology generated by the sets of the form $[F, U] = \{G \in \mathcal{F}[X]: F \subset G \subset U\}$, where $F \in \mathcal{F}[X]$ and U is an open neighborhood of F in X. E. K. van Douwen pointed out that $\mathcal{F}[X]$ is a zero-dimensional Tychonoff space [2]. For each $n \in N$, let $\mathcal{F}_n[X] = \{F \in \mathcal{F}[X]: \operatorname{card}(F) \leq n\}$. We consider $\mathcal{F}_n[X]$ with the subspace topology of $\mathcal{F}[X]$ for each $n \in N$.

T. Przymusiński characterized the paracompactness (resp. hereditary paracompactness) of Pixley-Roy hyperspaces as follows [6, Theorems 2.1, 2.2].

THEOREM 1. *The following conditions are equivalent*:

(a) $\mathcal{F}[X]$ *is paracompact*;

(b) $\mathcal{F}[X]^n$ *is paracompact for each* $n \in N$;

(c) *for every nonempty finite subset* F *of* X *one can choose an open neighborhood* U_F *so that the inclusions* $F \subset U_H$ *and* $H \subset U_F$ *imply* $F \cap H \neq \emptyset$.

THEOREM 2. *The following conditions are equivalent.*

(a) $\mathcal{F}[X]$ *is hereditarily paracompact*;

(b) $\mathcal{F}[X]^n$ *is hereditarily paracompact for each* $n \in N$;

(c) *for every nonempty finite subset* F *of* X *one can choose an open neighborhood* U_F *so that the inclusions* $F \subset U_H$ *and* $H \subset U_F$ *imply* $F \subset H$ *or* $H \subset F$.

So T. Przymusiński raised the following problem [6, Problem 2].

Problem. Suppose that $\mathcal{F}[X]$ is paracompact. Is $\mathcal{F}[X^2]$ paracompact?

The purpose of this paper is to answer this problem affirmatively. Furthermore we will obtain the hereditarily paracompact case. We will prove the following.

THEOREM 3. *Suppose that* $\mathcal{F}[X]$ *is paracompact* (*resp. hereditarily paracompact*). *Then* $\mathcal{F}[X^2]$ *is paracompact* (*resp. hereditarily paracompact*).

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Let $\mathcal{W}$ be a collection of subsets of $\mathfrak{F}[X]$. We denote $\bigcup\{W: W \in \mathcal{W}\}$ by $\mathcal{W}^*$. By $X \oplus Y$ we mean the topological sum of spaces X and Y. For undefined terminology, the reader is referred to [3]. For Pixley-Roy hyperspaces, the reader is referred to [2] and [6].

2. Proof of Theorem 3. The following lemma was proved by the author in [8].

LEMMA. *Let $\mathcal{Q}$ be an open-and-closed subset of $\mathfrak{F}[X]$ such that $\mathcal{Q} \supset \mathfrak{F}_n[X]$. For each $F \in \mathfrak{F}_{n+1}[X] - \mathcal{Q}$, let $\{W_F(x): x \in F\}$ be a collection of open subsets of X such that*:

(1) $x \in W_F(x)$ *for each* $x \in F$,

(2) $[F, \bigcup\{W_F(x): x \in F\}] \cap \mathcal{Q} = \varnothing$, *and*

(3) $[G, \bigcup\{W_F(x): x \in G\}] \subset \mathcal{Q}$ *for each* $G \subset F$ *such that* $\operatorname{card}(G) < n$.

If $[F, \bigcup\{W_F(x): x \in F\}] \cap [H, \bigcup\{W_H(y): y \in H\}] \neq \varnothing$ *for some F and H in $\mathfrak{F}_{n+1}[X] - \mathcal{Q}$, then each $W_F(x)$ contains exactly one element of H.*

PROOF OF THEOREM 3. Suppose that $\mathfrak{F}[X]$ is paracompact. Then there exists a collection $\mathfrak{U} = \{U_F: F \in \mathfrak{F}[X]\}$ of open subsets of X satisfying the condition (c) of Theorem 1. Let

$$\mathcal{W}_1 = \{[\{(x,y)\}, U_{\{x\}} \times U_{\{y\}}]: (x,y) \in X^2\}.$$

Suppose that

$$[\{(x,y)\}, U_{\{x\}} \times U_{\{y\}}] \cap [\{(z,w)\}, U_{\{z\}} \times U_{\{w\}}] \neq \varnothing$$

for some (x,y), $(z,w) \in X^2$. Then $(x,y) \in U_{\{z\}} \times U_{\{w\}}$ and $(z,w) \in U_{\{x\}} \times U_{\{y\}}$. Hence $x \in U_{\{z\}}, y \in U_{\{w\}}, z \in U_{\{x\}}$ and $w \in U_{\{y\}}$. Hence $x = z$ and $y = w$ by the property of $\mathfrak{U}$. Hence $(x,y) = (z,w)$. Hence $\mathcal{W}_1$ is pairwise disjoint in $\mathfrak{F}[X^2]$. We have already obtained a collection $\{\mathcal{W}_i: i = 1, \ldots, n\}$ of pairwise disjoint families of basic open subsets of $\mathfrak{F}[X^2]$ such that:

(3) $\bigcup_{i=1}^{j} \mathcal{W}_i^* \supset \mathfrak{F}_j[X^2]$ for each $j < n$, and

(4_n) $\mathcal{W}_i^* \cap \mathcal{W}_j^* = \varnothing$ for $i, j < n$ and $i \neq j$.

Let $\mathcal{Q} = \bigcup_{i=1}^{n} \mathcal{W}_i^*$. Then $\mathcal{Q}$ is a closed subset of $\mathfrak{F}[X^2]$ by Lemma 1 in [8] (or Lemma 2.3 in [6]). For each

$$(\dagger) \qquad F = \{(x_1,y_1), \ldots, (x_n,y_n), (x_{n+1},y_{n+1})\} \in \mathfrak{F}_{n+1}[X^2] - \mathcal{Q},$$

let $F_1 = \{x_1, \ldots, x_n, x_{n+1}\}$ and $F_2 = \{y_1, \ldots, y_n, y_{n+1}\}$. For each $x \in F_1$, let $F_1^x = F_1 - \{x\}$. For each $y \in F_2$, let $F_2^y = F_2 - \{y\}$. For each F in ($\dagger$), there exist open neighborhoods $W_F(x_i)$ and $W_F(y_i)$ of x_i and y_i, respectively, $i = 1, \ldots, n, n+1$, such that:

(5) $[F, \bigcup_{i=1}^{n+1} W_F(x_i) \times W_F(y_i)] \cap \mathcal{Q} = \varnothing$;

(6) for every subset G of F such that $\operatorname{card}(G) < n$, $[G, \bigcup_{(x,y)\in G} W_F(x) \times W_F(y)] \subset \mathcal{Q}$;

(7) for every subset H of F_1, $\bigcup_{x\in H} W_F(x) \subset U_H$;

(8) for every subset I of F_2, $\bigcup_{y\in I} W_F(y) \subset U_I$;

(9) for each $x \in F_1$, $W_F(x) \cap F_1^x = \varnothing$; and

(10) for each $y \in F_2$, $W_F(y) \cap F_2^y = \varnothing$.

For each F in (†) let

$$W(F) = \bigcup_{i=1}^{n+1} W_F(x_i) \times W_F(y_i).$$

Let

$$\mathcal{W}_{n+1} = \{[F, W(F)]: F \in \mathcal{F}_{n+1}[X^2] - \mathcal{Q}\}.$$

We will prove that $\mathcal{W}_{n+1}$ is pairwise disjoint in $\mathcal{F}[X^2]$. Suppose that $[F, W(F)] \cap [G, W(G)] \neq \varnothing$ for some

$$F = \{(x_1, y_1), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1})\},$$
$$G = \{(z_1, w_1), \ldots, (z_n, w_n), (z_{n+1}, w_{n+1})\} \in \mathcal{F}_{n+1}[X^2] - \mathcal{Q}.$$

Then $F \subset W(G)$ and $G \subset W(F)$. By the Lemma, $W_F(x_1) \times W_F(y_1)$ contains an element of G. So we can suppose $(z_1, w_1) \in W_F(x_1) \times W_F(y_1)$ (if necessary, by reordering the (z_i, w_i)'s). We will prove that $(x_1, y_1) \in W_G(z_1) \times W_G(w_1)$. Suppose that $(x_1, y_1) \notin W_G(z_1) \times W_G(w_1)$. By the Lemma, $W_G(z_1) \times W_G(w_1)$ contains an element of F. So we can suppose $(x_2, y_2) \in W_G(z_1) \times W_G(w_1)$. Then $(z_1, w_1) \notin W_F(x_2) \times W_F(y_2)$. For if not, suppose $(z_1, w_1) \in W_F(x_2) \times W_F(y_2)$. By the Lemma,

$$G - \{(z_1, w_1)\} \subset \bigcup_{i=3}^{n+1} W_F(x_i) \times W_F(y_i).$$

But $\operatorname{card}(F - \{(x_1, y_1), (x_2, y_2)\}) = n - 1$ and $\operatorname{card}(G - \{(z_1, w_1)\}) = n$. Hence there exists some i $(3 < i < n + 1)$ such that $W_F(x_i) \times W_F(y_i)$ contains two distinct elements of G, which contradicts the Lemma. Hence $(z_1, w_1) \notin W_F(x_2) \times W_F(y_2)$. We can suppose that $(z_2, w_2) \in W_F(x_2) \times W_F(y_2)$. By the same reason, $(x_2, y_2) \notin W_G(z_2) \times W_G(w_2)$. So we can suppose that $(x_3, y_3) \in W_G(z_2) \times W_G(w_2)$. We continue this process repeatedly. Then we obtain subsets $B = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ and $C = \{(z_1, w_1), \ldots, (z_m, w_m)\}$, $m < n + 1$, of F and G, respectively, such that:

(11) $(z_i, w_i) \in W_F(x_i) \times W_F(y_i)$ for each $i < m$, and

(12) $(x_{i+1}, y_{i+1}) \in W_G(z_i) \times W_G(w_i)$ for each $i < m - 1$ and $(x_1, y_1) \in W_G(z_m) \times W_G(w_m)$.

Let $H = \{x_1, \ldots, x_m\}$ and $I = \{z_1, \ldots, z_m\}$. Then

$$H \cup I \subset \bigcup_{i=1}^{m} W_F(x_i) \cap \bigcup_{i=1}^{m} W_G(z_i).$$

By (7), $H \cup I \subset U_H \cap U_I$. By the property of $\mathfrak{U}$, $H \cap I \neq \varnothing$. Hence $x_i = z_j$ for some $i, j < m$. By (12), $(x_i, y_i) \in W_G(z_{i-1}) \times W_G(w_{i-1})$ (if $i = 1$, then $(x_1, y_1) \in W_G(z_m) \times W_G(w_m)$). Suppose that $x_i \neq z_{i-1}$. Then $z_{i-1} \neq z_j$. By (9), $W_G(z_{i-1}) \not\ni z_j$, which is a contradiction. Hence $x_i = z_{i-1}$. Since $(z_{i-1}, w_{i-1}) \in W_F(x_{i-1}) \times W_F(y_{i-1})$, we similarly obtain that $x_{i-1} = z_{i-1}$. By (11) and (12), we can continue this process repeatedly. Then we obtain $x_1 = z_1 = \cdots = x_m = z_m$. Similarly we obtain $y_1 = w_1 = \cdots = y_m = w_m$. Hence $(x_1, y_1) = \cdots = (x_m, y_m)$, which contradicts the fact that $F \in \mathcal{F}_{n+1}[X^2]$. Hence $(x_1, y_1) \in W_G(z_1) \times W_G(w_1)$. By (7) and (8), we obtain $x_1 = z_1$ and $y_1 = w_1$. Hence $(x_1, y_1) = (z_1, w_1)$. Similarly we obtain $(x_i, y_i) = (z_i, w_i)$ for each $i = 2, \ldots, n, n+1$ (if necessary, by reordering the (z_i, w_i)'s, $i = 2, \ldots, n, n+1$). Hence $F = G$. Hence $\mathfrak{W}_{n+1}$ is pairwise disjoint in $\mathfrak{F}[X^2]$. Clearly $\{\mathfrak{W}_i : i = 1, \ldots, n, n+1\}$ satisfies conditions (3_{n+1}) and (4_{n+1}). Inductively we obtain a sequence $\{\mathfrak{W}_n : n \in N\}$ of pairwise disjoint families of basic open subsets of $\mathfrak{F}[X^2]$ such that:

(13) $\bigcup\{\mathfrak{W}_n : n \in N\}$ covers $\mathfrak{F}[X^2]$, and

(14) $\mathfrak{W}_i^* \cap \mathfrak{W}_j^* = \varnothing$ for $i, j \in N$ and $i \neq j$.

Let $\mathfrak{W} = \bigcup\{\mathfrak{W}_n : n \in N\}$. We denote $\mathfrak{W}$ by $\{[F_s, W(F_s)] : s \in S\}$. For each $F \in \mathfrak{F}[X^2]$, there exists exactly one $s \in S$ such that $F \in [F_s, W(F_s)]$. Define $V_F = W(F_s)$. Let $\mathfrak{V} = \{V_F : F \in \mathfrak{F}[X^2]\}$. It is easily proved that the collection $\mathfrak{V}$ satisfies the condition (c) of Theorem 1. Hence $\mathfrak{F}[X^2]$ is paracompact.

Next we consider the hereditarily paracompact case. Suppose that $\mathfrak{F}[X]$ is hereditarily paracompact. Then there exists a collection $\mathfrak{U} = \{U_F : F \in \mathfrak{F}[X]\}$ of open subsets of X satisfying the condition (c) of Theorem 2. For each $F = \{(x_1, y_1), \ldots, (x_n, y_n)\} \in \mathfrak{F}[X^2]$, where $n = \operatorname{card}(F)$, let $\tilde{F} = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. For each $z \in \tilde{F}$, let $\tilde{F}_z = \tilde{F} - \{z\}$. For each $z \in \tilde{F}$, let $W_F(z) = U_{\tilde{F}} - \tilde{F}_z$. Then $\bigcup_{z \in \tilde{F}} W_F(z) = U_{\tilde{F}}$. For each $F = \{(x_1, y_1), \ldots, (x_n, y_n)\} \in \mathfrak{F}[X^2]$, where $n = \operatorname{card}(F)$, let $W(F) = \bigcup_{i=1}^n W_F(x_i) \times W_F(y_i)$. Let $\mathfrak{W} = \{W(F) : F \in \mathfrak{F}[X^2]\}$. It suffices to prove that the collection $\mathfrak{W}$ satisfies the condition (c) of Theorem 2. Suppose that $F \subset W(H)$ and $H \subset W(F)$ for some $F = \{(x_1, y_1), \ldots, (x_n, y_n)\}$, $H = \{(z_1, w_1), \ldots, (z_m, w_m)\} \in \mathfrak{F}[X^2]$, where $n = \operatorname{card}(F)$ and $m = \operatorname{card}(H)$. Then $\tilde{F} \subset \bigcup_{q \in \tilde{H}} W_H(q)\ (= U_{\tilde{H}})$ and $\tilde{H} \subset \bigcup_{p \in \tilde{F}} W_F(p)\ (= U_{\tilde{F}})$. By the property of $\mathfrak{U}$, $\tilde{F} \subset \tilde{H}$ or $\tilde{H} \subset \tilde{F}$. Without loss of generality, we can suppose that $\tilde{F} \subset \tilde{H}$. Since $F \subset W(H)$, we can suppose that $(x_1, y_1) \in W_H(z_1) \times W_H(w_1)$ (if necessary, by reordering the (z_j, w_j)'s). Suppose that $x_1 \neq z_1$. Since $\tilde{F} \subset \tilde{H}$, $x_1 \in \tilde{H}$. By the definition of $W_H(z_1)$, $W_H(z_1) \not\ni x_1$. But $(x_1, y_1) \in W_H(z_1) \times W_H(w_1)$. This contradiction shows that $x_1 = z_1$. Similarly $y_1 = w_1$. Hence $(x_1, y_1) = (z_1, w_1)$. Similarly we obtain $(x_i, y_i) = (z_i, w_i)$ for each $i = 2, \ldots, n$ (if necessary, by reordering the (z_j, w_j)'s, $j = 2, \ldots, m$). Hence $F \subset H$. Hence $\mathfrak{W}$ satisfies the condition (c) of Theorem 2. Hence $\mathfrak{F}[X^2]$ is hereditarily paracompact.

3. Applications.

THEOREM 4. *Suppose that $\mathfrak{F}[X]$ and $\mathfrak{F}[Y]$ are paracompact. Then $\mathfrak{F}[X \times Y]$ is paracompact.*

PROOF. In the proof of Theorem 3, we in fact prove this theorem. We will give another proof. Let $Z = X \oplus Y$. By Proposition 3.1 in [6], $\mathfrak{F}[Z]$ is paracompact. Hence $\mathfrak{F}[Z^2]$ is paracompact by Theorem 3. Since $X \times Y$ is a subspace of Z^2, $\mathfrak{F}[X \times Y]$ is a closed subspace of $\mathfrak{F}[Z^2]$ (see Proposition 1.2 in [6]). Hence $\mathfrak{F}[X \times Y]$ is paracompact.

REMARK 1. There exist spaces X and Y such that $\mathfrak{F}[X]$ and $\mathfrak{F}[Y]$ are hereditarily paracompact, but $\mathfrak{F}[X \oplus Y]$ and $\mathfrak{F}[X \times Y]$ are not (see [1, Theorem 2.5], [4, Theorem 2.8], and [6, Example 6.6]).

THEOREM 5. *The following conditions are equivalent.*

(a) $\mathfrak{F}[X]$ *is paracompact*;

(b) $\mathfrak{F}[X^2]$ *is paracompact*;

(c) $\mathfrak{F}[X^m]$ *is paracompact for each* $m \in N$;

(d) $\mathfrak{F}[X^m]^n$ *is paracompact for each* $m, n \in N$.

PROOF. By Theorem 1, conditions (c) and (d) are equivalent. The implication (c) $\to$ (b) is obvious.

(b) $\to$ (a). Since X^2 contains a copy of X, we can consider $\mathfrak{F}[X]$ a closed subspace of $\mathfrak{F}[X^2]$ (see Proposition 1.2 in [6]). Hence this implication follows obviously.

(a) $\to$ (c). Suppose that $\mathfrak{F}[X]$ is paracompact. By Theorem 4, it is easily proved by induction that $\mathfrak{F}[X^m]$ is paracompact for each $m \in N$.

REMARK 2. Let D be a two point discrete space. Clearly $\mathfrak{F}[D]$ is paracompact. But $\mathfrak{F}[D^\omega]$ is not normal, where ω is the first infinite ordinal (see [7]). Therefore we obtain that Theorem 5 cannot be sharpened to assert that $\mathfrak{F}[X^\omega]$ is paracompact.

THEOREM 6. *The following conditions are equivalent.*

(a) $\mathfrak{F}[X]$ *is hereditarily paracompact*;

(b) $\mathfrak{F}[X^2]$ *is hereditarily paracompact*;

(c) $\mathfrak{F}[X^m]$ *is hereditarily paracompact for each* $m \in N$;

(d) $\mathfrak{F}[X^m]^n$ *is hereditarily paracompact for each* $m, n \in N$.

PROOF. It suffices to prove the implication (a) $\to$ (c). Suppose that $\mathfrak{F}[X]$ is hereditarily paracompact. By Theorem 3, $\mathfrak{F}[X^{2^n}]$ is hereditarily paracompact for each $n \in N$. For each $m \in N$, there exists an $n \in N$ such that $m < 2^n$. Since X^{2^n} contains a copy of X^m, we can consider $\mathfrak{F}[X^m]$ a closed subspace of $\mathfrak{F}[X^{2^n}]$ (see Proposition 1.2 in [6]). Hence $\mathfrak{F}[X^m]$ is hereditarily paracompact. Hence the implication (a) $\to$ (c) follows.

By Theorems 4 and 5, we obtain the following.

THEOREM 7. *Let* $\{X_i: i = 1, \ldots, s\}$ *be a finite collection of spaces such that* $\mathfrak{F}[X_i]$ *is paracompact for each* $i = 1, \ldots, s$. *Then* $\mathfrak{F}[X_1^{m_1} \times \cdots \times X_s^{m_s}]^n$ *is paracompact for* $m_1, \ldots, m_s, n \in N$.

E. K. van Douwen proved that $\mathfrak{F}[X]$ is a Moore space if and only if X is first countable [2, Proposition 2.5]. It is well known that for a paracompact Hausdorff space X, X is metrizable if and only if X is a Moore space (see [3, 5.4.1]). Therefore we obtain the following.

THEOREM 8. *The following conditions are equivalent.*

(a) $\mathfrak{F}[X]$ *is metrizable*;

(b) $\mathfrak{F}[X^2]$ *is metrizable*;

(c) $\mathfrak{F}[X^m]$ *is metrizable for each* $m \in N$;

(d) $\mathfrak{F}[X^m]^n$ *is metrizable for each* $m, n \in N$.

THEOREM 9. *Let* $\{X_i: i = 1, \ldots, s\}$ *be a finite collection of spaces such that* $\mathfrak{F}[X_i]$ *is metrizable for each* $i = 1, \ldots, s$. *Then* $\mathfrak{F}[X_1^{m_1} \times \cdots \times X_s^{m_s}]^n$ *is metrizable for* $m_1, \ldots, m_s, n \in N$.

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