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ON SEMIPRIME RINGS OF BOUNDED INDEX

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ABSTRACT. A ring R is of bounded index (of nilpotency) if there is an integer $n \geq 1$ such that $x^n = 0$ whenever $x \in R$ is nilpotent. The least such positive integer is the index of R. We show that a semiprime ring R has index $\leq n$ if and only if R is a subdirect product of prime rings of index $\leq n$.

A ring R is of *bounded index* if there is an integer $n \geq 1$ such that $x^n = 0$ whenever x is a nilpotent element of R. The least such positive integer is called the *index of* R, and we denote it by $i(R)$. V. A. Andrunakievič and Ju. M. Rjabuhin [**1**], and, independently, P. N. Stewart [**6**] have shown that a ring of index 1 (i.e. a ring with no nonzero nilpotent elements) is a subdirect product of integral domains. In this article we will show that a semiprime ring of index n is a subdirect product of prime rings each of index at most n.

For a subset X of a ring R we let $r(X)$ denote the right annihilator of X, while $l(X)$ denotes the left annihilator of X. If T is an ideal of a semiprime ring R, then it is easily seen that $l(T) = r(T)$; hence we write $\operatorname{ann}(T)$ for this two-sided ideal of R.

We begin by adding one more equivalence to the following result of J. Hannah [**4**, Proposition 2].

LEMMA 1. *Let R be a semiprime ring, n a positive integer. Then the following statements are equivalent.*

(a) $i(R) \leq n$.

(b) *If $X_1, \ldots, X_n \subseteq R$ such that $X_iX_j = 0$ whenever $i \geq j$, then $X_1X_2 \ldots X_n = 0$.*

(c) *If $X subseteq R$, then $r(X^n) = r(X^{n+1})$.*

(d) *For each $x \in R$, $r(x^n) = r(x^{n+1})$.*

LEMMA 2. *Let R be a semiprime ring and n a positive integer. Then $i(R) \leq n$ if and only if*

$$(*) \qquad \text{for each } x \in R \text{ and each ideal } T \text{ of } R,\ r(Tx^n) = r(Tx^{n+1}).$$

PROOF. If $(*)$ holds, then letting $T = R$ we get $r(Rx^n) = r(Rx^{n+1})$. However R is semiprime so $r(Ra) = r(a)$ for all $a \in R$. Hence (d) of Lemma 1 is satisfied so that $i(R) \leq n$. Thus assume $i(R) \leq n$. We shall use Lemma 1(b) to establish $(*)$. Proceeding as in Hannah's proof, we let $X_i = r(Tx^i)Tx^i$ for $i = 1, 2, \ldots, n$. Then for $i \geq j$ we have $X_iX_j = 0$ so we must have

$$\begin{aligned} 0 &= X_1X_2 \cdots X_n = T \cdot X_1X_2 \cdots X_n \\ &= [T \cdot r(Tx)][Tx \cdot r(Tx^2)][Tx^2 \cdot r(Tx^3)] \cdots [Tx^{n-1} \cdot r(Tx^n)]Tx^n. \end{aligned}$$

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Now for each $i \geq 0$ (with $Tx^0 = T$) we have $Tx^n \cdot r(Tx^{n+1}) = Tx^i \cdot x^{n-i} r(Tx^{n+1})$ and $x^{n-i} \cdot r(Tx^{n+1}) \subseteq r(Tx^{i+1})$. Hence $Tx^n \cdot r(Tx^{n+1}) \subseteq Tx^i \cdot r(Tx^{i+1})$ for $i \geq 0$. But then $[Tx^n \cdot r(Tx^{n+1})]^{n+1} = 0$ so $Tx^n \cdot r(Tx^{n+1}) = 0$ since R is semiprime. Thus $r(Tx^{n+1}) \subseteq r(Tx^n)$ and equality follows.

LEMMA 3. *Assume R is a semiprime ring of index n and T is an ideal of R. If $K = \operatorname{ann}(T)$ then $i(R/K) \leq n$.*

PROOF. Since $K = \operatorname{ann}(T)$, R/K is a semiprime ring. Let $\bar{R} = R/K$ and let $\bar{a} = a + K \in \bar{R}$. If $\bar{w} \in r(\bar{a}^{n+1})$, then $a^{n+1}w \in K$ so $Ta^{n+1}w = 0$. Since $r(Ta^{n+1}) = r(Ta^n)$, we have $a^n w \in K$ and so $\bar{w} \in r(\bar{a}^n)$. By Lemma 1(c), $i(R/K) \leq n$.

THEOREM 1. *Let R be a semiprime ring of index n. Then R is a subdirect product of prime rings of index $\leq n$.*

PROOF. We must show that for each $a \in R$, $a \neq 0$, there is a prime ideal P with $a \notin P$ and $i(R/P) \leq n$. Since R is semiprime and $a \neq 0$, there is an *m-system* containing a [**5**, Chapter 4]. Specifically let $a_1 = a$; then $a_1 R a_1 \neq 0$ implies $a_1 b_1 a_1 = a_2 \neq 0$ for some $b_1 \in R$. Inductively we obtain sequences $\{b_i\}$, $\{a_i\}$ with $a_{i+1} = a_i b_i a_i \neq 0$ and $\mathcal{M} = \{a_i\}$ is the desired *m-system*. Using Zorn's Lemma, the set $\{A \mid A \text{ an ideal of } R,\ \mathcal{M} \cap A = \emptyset \text{ and } i(R/A) \leq n\}$ has a maximal element P which we claim is a prime ideal of R. To show that P is a prime ideal, we first note that P is a semiprime ideal of R. For if $P \subseteq W$ and $W^2 \subseteq P$, then $i(R/W) \leq n$ since $i(R/P) \leq n$. Now if $P \subseteq W$, then $\mathcal{M} \cap W \neq \emptyset$ so $a_i \in W$ for some $i \geq 1$. But then $a_i R a_i \subseteq W^2 \subseteq P$, so $a_{i+1} \in P$, a contradiction. Thus P is indeed a semiprime ideal of R. By passing to R/P we assume that $P = 0$. If R is semiprime but not prime, we have nonzero ideals A, B with $A = \operatorname{ann}(B)$ and $B = \operatorname{ann}(A)$. Then by Lemma 3, $i(R/A) \leq n$ and $i(R/B) \leq n$. Hence $\mathcal{M} \cap A \neq \emptyset$ and $\mathcal{M} \cap B \neq \emptyset$. But if $a_i \in \mathcal{M} \cap A$, $a_j \in \mathcal{M} \cap B$ with $j \geq i$, then $a_j \in A \cap B = 0$, a contradiction. Thus P is a prime ideal, as desired.

As a consequence we have the index 1 result.

THEOREM 2 (ANDRUNAKIEVIČ AND RJABUHIN; STEWART). *A ring R has no zero nilpotent elements if and only if R is a subdirect product of integral domains.*

In addition we deduce

THEOREM 3. *Let R be a ring of index n, and let N denote the prime radical of R. Then $N = \bigcap\{P \mid P \text{ is a prime ideal, } i(R/P) \leq n\}$.*

PROOF. Since N is a nil ideal, $i(R/N) \leq n$ and R/N is semiprime so the statement follows.

Hannah also establishes in [**4**, Proposition 5] that a prime ring has index $\leq n$ if and only if each chain of left (right) annihilators has at most n proper inclusions. Thus a prime ring of bounded index has ACC on left and right annihilators. As Hannah points out, however, semiprime rings of index n need not have the ACC on left or right annihilators. (An infinite direct product of fields provides an appropriate example.) There is, however, a class of semiprime rings for which bounded index yields the ACC on left and right annihilators. A semiprime ring with unit is (right) *strongly semiprime* if each faithful ideal contains a finitely generated left ideal whose right annihilator is zero. A strongly prime ring is a prime ring

which is strongly semiprime. For properties of these rings see [**2, 3**]. In particular D. Handelman shows [**3**, Theorem 1] that a ring R is strongly semiprime if and only if R is a finite subdirect product of strongly prime rings. Lemma 3 is instrumental in showing

THEOREM 4. *A ring R is strongly semiprime of index $\leq n$ if and only if R is a finite subdirect product of strongly prime rings of index $\leq n$.*

PROOF. According to [**3**, Theorem 1] the prime rings are obtained by taking R/P, where P is maximal among the annihilators of two-sided ideals of R. If R has index n, then, by Lemma 3, $i(R/P) \leq n$. The converse is evident.

Using [**4**, Proposition 5] yields

THEOREM 5. *If R is a strongly semiprime ring of index n then R has ACC on left and right annihilators.*

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