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ON FUNCTIONS IN THE BALL ALGEBRA

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ABSTRACT. We show that there exists a function in a ball algebra such that almost every slice function has a series of Taylor coefficients divergent with every power $p < 2$.

In §7.2 of [3] W. Rudin gives some examples of boundary behavior of holomorphic functions in complex balls of dimension 2 and 3. It was observed in [4, Remark 1.10], that using Theorem 1.2 of [4] such examples can be constructed in arbitrary dimension. In the present note we further pursue this idea. It is well known that in one variable there exists a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic for $|z| \leq 1$ such that $\sum |a_n|^p = \infty$ for all $p < 2$. Our theorem generalizes this fact to functions of several variables.

In this note, B will always denote the unit ball in the complex d-dimensional space C^d, S will stand for the unit sphere and σ for the rotation invariant probability measure on S. $A(B)$ will denote the ball algebra of all functions analytic in B and continuous in $\overline{B}$. For $0 < p \leq \infty$, $\|f\|_p$ denotes $(\int_S |f(\varsigma)|^p d\sigma(\varsigma))^{1/p}$. If f is a holomorphic function in B then it has a unique homogeneous expansion as $f = \sum_{n=0}^{\infty} f_n$, f_n is an analytic polynomial homogeneous of degree n.

It was shown in [4, Theorem 1.2], that there exist polynomials (p_n) homogeneous of degree n on B such that

$$(*) \qquad \|p_n\| = 1 \quad \text{and} \quad \|p_n\|_\infty \leq \chi.$$

(χ depends only on the dimension of the ball, it can be taken $\chi = 2^d/\sqrt{\pi}$.) Those polynomials will be crucial in our further considerations. In particular, we will use the following inequality (cf. [4, Proposition 1.6])

$$(**) \qquad \left\| \sum_{k=0}^{\infty} \alpha_n p_{2^n} \right\|_p \sim \left(\sum_{k=0}^{\infty} |\alpha_n|^2 \right)^{1/2}, \quad 0 < p < \infty.$$

We will also use the fact that Cauchy Integral $C[\mu]$ maps continuously measures on S into all $H_p(B)$, $p < 1$ (cf. [3, 6.2]).

The book [3] is an excellent source of information about the function theory in B.

PROPOSITION. *The operator $T: A(B) \to \ell_2$ defined by $T(f) = (\langle f, p_{2^n} \rangle)_{n=0}^{\infty}$ is a surjective map.*

PROOF. This map is clearly continuous. By duality it is enough to show that $T^*: \ell_2 \to A(B)^*$ is an isomorphic embedding i.e. $\|T^*(\alpha_n)\|_{A(B)^*} \geq C(\sum |\alpha_n|^2)^{1/2}$.

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One easily checks that $T^*(\alpha_n)(f) = \langle f, \sum_{n=0}^{\infty} \alpha_n p_{2^n}\rangle$. Let μ be a measure on S such that, for $f \in A(B)$, $T^*(\alpha_n)(f) = \int f\, d\mu$ and $\|\mu\| = \|T^*(\alpha_n)\|$. Then obviously $C[\overline{\mu}]$, the Cauchy Integral of $\overline{\mu}$, equals $\sum_{n=0}^{\infty} \alpha_n p_{2^n}$, so (by properties of $C[\cdot]$ and $(**)$)

$$\|T^*(\alpha_n)\| = \|\mu\| = \|\overline{\mu}\| \geq C\|C[\overline{\mu}]\|_{1/2} = C\left\|\sum_{n=0}^{\infty} \alpha_n p_{2^n}\right\|_{1/2} \geq C\left(\sum_{n=0}^{\infty} |\alpha_n|^2\right)^{1/2}.$$

COROLLARY. *There exists a constant C such that for every sequence $(\alpha_n)_{n=0}^{\infty}$ with $\alpha_n = 0$ for $n \leq K$ and $n \geq N$ (for some integers K and N) there exists $f \in A(B)$ such that*

$$\|f\|_\infty \leq C\left(\sum_{n=0}^{\infty} |\alpha_n|^2\right)^{1/2}, \quad Tf = (\alpha_n),$$

and the homogeneous expansion of f has a form

$$\sum_{s=2^K+1}^{2^N-1} f_s.$$

PROOF. Let β be a piecewise linear function given by

$$\beta(t) = \begin{cases} 0 & \text{if } t \leq 2^K \text{ or } t \geq 2^N, \\ 1 & \text{if } 2^{K+1} \leq t \leq 2^{N-1}, \\ \text{linear} & \text{otherwise.} \end{cases}$$

It follows from the classical de la Valleé-Poussin theorem that the map $f = \sum_{n=0}^{\infty} f_n \mapsto \sum_{n=0}^{\infty} \beta(n) \cdot f_n$ maps $A(B)$ continuously into $A(B)$, and has norm bounded by an absolute constant. This observation and the proposition proves the corollary.

THEOREM. *There exists a function $f \in A(B)$ with the homogeneous expansion $f = \sum_{n=0}^{\infty} f_n$ such that for every $p < 2$ and σ-almost all $\varsigma \in S$, $\sum_{n=0}^{\infty} |f_n(\varsigma)|^p = \infty$ i.e., almost every slice function of f has a series of Taylor coefficients divergent with every power < 2.*

PROOF. Let us denote $2^{2^{r+k}}$ by $\gamma(r,k)$. Repeated use of the corollary gives a sequence of functions $g^r \in A(B)$ such that

(1) the homogeneous expansion of g^r has form

$$g^r = \sum_{\gamma(r,0)+1}^{\gamma(r+1,0)-1} g_k^r;$$

(2) $\|g^r\|_\infty \leq C$;

(3) $\langle g^r, p_{\gamma(r,k)}\rangle = 2^{-r/2}$, $k = 1, 2, \ldots, 2^r - 1$.

The properties (1)–(3) imply that there exist two constants α, β such that for every r and p

(4) $\sum_k |g_k^r(\varsigma)|^p \geq \alpha 2^{r(1-p/2)}$ on a set $A_r \subset S$ with $\sigma(A_r) \geq \beta$.

PROOF OF (4). By (3) and $(*)$ we infer

(5) $\int |g^r_{\gamma(r,k)}| d\sigma \geq \chi^{-1} 2^{-r/2}$.

Since $C^2 \geq \|g^r\|_2^2 > \sum_{k=1}^{2^r-1} \|g^r_{\gamma(r,k)}\|_2^2$ we see that for at least 2^{r-1} of the functions $(g^r_{\gamma(r,k)})_{k=1}^{2^r-1}$ we have

(6) $\|g^r_{\gamma(r,k)}\|_2 \leq 2C2^{-r/2}$.

Using (5) for each k satisfying (6) we obtain a set B_k, $\sigma(B_k) \geq 16^{-1}C^{-2}\chi^{-2}$, such that $|g^r_{\gamma(r,k)}(\varsigma)| \geq 1/2\chi \cdot 2^{-r/2}$ for $\varsigma \in B_k$.

From this follows that for some α and β (independent of r) the set A_r of all points in S, which belong to at least $\alpha \cdot 2^r$ sets B_k, has measure $\geq \beta$. This clearly proves (4).

Using the "scrambling lemma" [3, p. 129] we find a sequence U_r of unitary maps such that almost every $\varsigma \in S$ belongs to infinitely many sets $U_r(A_r)$. We define the desired function by

(7) $f = \sum_{r=1}^{\infty} r^{-2} g^r \circ U_r^{-1}$.

By (1) we infer that the homogeneous expansion $\sum_{n=0}^{\infty} f_n$ for f actually equals $\sum_{r=1}^{\infty} \sum_{k=\gamma(r,0)+1}^{r(r+1,0)} r^{-2} g^r_k \circ U_r^{-1}$. Since $\|g^r \circ U_r^{-1}\|_\infty = \|g_r\|_\infty \leq C$ the series (7) is absolutely convergent and defines a function in $A(B)$. By (4), for $\varsigma \in U_r(A_r)$, we have $\sum_k |g^r_k \circ U_r^{-1}(\varsigma)|^p \geq \alpha 2^{r(1-p/2)}$

Given $\varsigma \in S$

$$\sum_{n=0}^{\infty} |f_n(\varsigma)|^p = \sum_{r=1}^{\infty} r^{-2} \sum_{k=\gamma(r,0)+1}^{r(r+1,0)-1} |g^r_k \circ U_r^{-1}(\varsigma)|^p \geq \sum_{r:\, \varsigma \in U_r(A_r)} \alpha r^{-2} 2^{r(1-p/2)}.$$

This sum is clearly infinite for every $p < 2$ and every ς belonging to infinitely many sets $U_r(A_r)$. This completes the proof.

REMARK 1. I do not know the answer to the following question: given $(\alpha_n) \in \ell_2$ does there exist $f = \sum f_k \in A(B)$ such that $f_{2^k} = \alpha_k p_{2^k}$? A positive answer to this question would imply the theorem.

REMARK 2. Our proof of the proposition is the same as the one given for the disc algebra in [2, Proposition 3.1]. In one variable there are also beautiful constructive proofs (e.g. in [1]). It would be interesting to have such a proof in our case.

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