NOMOGRAPHIC FUNCTIONS ARE NOWHERE DENSE

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ABSTRACT. A function $f$ of $n$ variables is nomographic if it can be represented in the format

$$f(x_1, \ldots, x_n) = h(\phi_1(x_1) + \cdots + \phi_n(x_n))$$

where the $\phi_i$ and $h$ are continuous. Every continuous function of $n$ variables has a representation as a sum of not more than $2n + 1$ nomographic functions [9]. This paper gives a constructive proof that the nomographic functions form a nowhere dense subset of the space $C[I^n]$.

Let $I = [-1,1]$ and $I^k = I \times I \times \cdots \times I$. The class $\mathcal{N}^k$ of nomographic functions of $k$ variables are those that on $I^k$ have a representation in the special format

$$(1) \quad f(x_1, x_2, \ldots, x_k) = h(\phi_1(x_1) + \phi_2(x_2) + \cdots + \phi_k(x_k))$$

where the $\phi_k$ and $h$ are real-valued continuous functions on $-\infty < t < \infty$. Interest in $\mathcal{N}^k$ revived when Kolmogorov used these functions in 1957 to settle Hilbert's 13th problem by showing that every continuous function on $I^k$ could be written as the sum of $2k + 1$ functions from $\mathcal{N}^k$. (See [9, 10].)

Formula (1) makes it natural to conjecture that $\mathcal{N}^k$ is a sparse subset of the space $C[I^k]$ of all real continuous functions on $I^k$, with the usual norm $\|g\| = \max|g(p)|$, as suggested in [2]. This does not conflict with the Kolmogorov result; for example, $[0,1]$ is the algebraic sum of two copies of a nowhere dense subset $E$. That $\mathcal{N}^k$ should be sparse is more evident when smoothness is required. If $f \in \mathcal{N}^2$ has component functions $h$, $\phi_1$, and $\phi_2$ which are in $C'''$, then $f$ must satisfy a third order PDE that is characteristic for $\mathcal{N}^2$. (See [4].) In addition, nowhere denseness is known to follow smoothness in a number of other studies of superposition classes. (See [6, 7, 8].) In the present paper, we present a direct constructive proof that $\mathcal{N}^k$ is nowhere dense in $C[I^k]$, with no differentiability restrictions on $h$ or $\phi_j$; indeed, we do not require that $h$ be continuous. We remark that $\mathcal{N}^2$ is not uniformly closed. (See [1 or 3].)

We prove a more general theorem, and then verify later that the requisite property is shared by the class $\mathcal{N}^k$.

Let $D$ be a compact set in $R^n$ with nonvoid interior, and $C[D]$ be the Banach space of real-valued continuous functions on $D$, with the uniform convergence norm $\|F\|_D = \max_{p \in D}|F(p)|$. Let $\mathcal{F}$ be a subset of $C[D]$ which we will prove is nowhere dense. The key requirement we need is the existence of special functions in $C[D]$ that fail to belong locally to the closure of $\mathcal{F}$.
THEOREM 1. Suppose that there is a point $p_0$ interior to $D$ such that for any real $c$, there exists $g \in C[D]$ with $g(p_0) = c$ such that for every compact neighborhood $V$ of $p_0$

\[ \inf_{f \in \mathcal{F}} \|f - g\|_V > 0. \]

Then, $\mathcal{F}$ is nowhere dense in $C[D]$.

PROOF. Let $U$ be any nonempty open set in $C[D]$; we will produce another nonempty open set $U_1 \subset U$ disjoint from $\mathcal{F}$. Choose $G_0 \in U$ and $r > 0$ such that $\|G - G_0\| < r$ implies $G \in U$. Let $c = G_0(p_0)$, and then use the hypothesis to select a special function $g$ in $C[D]$ obeying (2). Let $B$ be a closed ball in $D$, centered at $p_0$, such that

\[ |G_0(p) - c| < \frac{\varepsilon}{3} \quad \text{for all } p \in B, \]

and choose a smaller ball $B_0$, also centered at $p_0$, such that

\[ |g(p) - c| < \frac{\varepsilon}{3}, \quad p \in B_0. \]

Construct a continuous function $G_1$ defined on $B$ such that $G_1(p) = G_0(p)$ for $p$ on the boundary of $B$, $G_1(p) = g(p)$ on $B_0$, and obeying $|G_1(p) - c| < \frac{\varepsilon}{3}$ on $B$. Then, extend $G_1$ to all of $D$ by setting it equal to $G_0$ off $B$. Observe that $G_1$ is in $C[D]$ and agrees with $G_0$ except on a small neighborhood of $p_0$ where it has been modified to agree with the special function $g$ locally. If $p \in B$, we have

\[ |G_1(p) - G_0(p)| \leq |G_1(p) - c| + |c - G_0(p)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < r. \]

Accordingly, $\|G_1 - G_0\|_D < r$ and $G_1 \in U_1$.

By (2), choose $\delta > 0$ so that $\|f - g\|_{B_0} > \delta$ for all $f \in \mathcal{F}$, and take $U_1 = \{F \in U$ with $\|F - G_1\|_D < \frac{\delta}{2}\}$. Then, if $F \in U_1$ and $f \in \mathcal{F}$,

\[ \|f - F\|_D \geq \|f - g\|_{B_0} - \|F - G_1\|_{B_0} \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}. \]

Thus, $\mathcal{F}$ is nowhere dense in $C[D]$.

We now use this to show that $\mathcal{N}^2$ is nowhere dense in $C[I^2]$. Take $p_0 = (0,0)$; since $\mathcal{N}^2$ is closed under addition of constants, we do not need to retain the condition in (2) that $g(p_0) = c$. For any $r > 0$, let $V_r$ be the compact neighborhood of $p_0$ consisting of those $(x, y)$ with $|x| \leq r$, $|y| \leq r$.

THEOREM 2. The special function $g(x, y) = x^2 + xy + y^2 + 2x + y$ has the property that

\[ r^2 > \inf_{f \in \mathcal{N}^2} \|f - g\|_{V_r} > r^3/10 \]

for all $r < .01$.

We begin the proof of this by quoting one of the characterization theorems for nomographic functions obtained in [3], modifying it to match the notation and needs of the present paper. (See Theorem 12, p. 293.)
Let $g$ be of class $C'$ on the set $V_r$, and suppose that $g_x$ and $g_y$ are bounded below by $\sigma > 0$. Let $\varepsilon < r\sigma/12$ and suppose that the distance in the space $C[V_r]$ between $g$ and the set $\mathcal{N}^2$ is less than $\varepsilon$. Then, one of the following systems of inequalities must be solvable.

(i) For some choice of $x_i$ in $[-r,r]$,

\[
|g(x_1, -r) - g(-r, 0)| < 2\varepsilon,
\]

\[
|g(x_2, -r) - g(-r, r)| < 2\varepsilon,
\]

\[
|g(x_1, r) - g(x_2, 0)| < 2\varepsilon.
\]

(ii) For some choice of $y_i$ in $[-r, r]$,

\[
|g(-r, y_1) - g(0, -r)| < 2\varepsilon,
\]

\[
|g(-r, y_2) - g(r, -r)| < 2\varepsilon,
\]

\[
|g(r, y_1) - g(0, y_2)| < 2\varepsilon.
\]

If we apply this general result to the special function $g(x, y)$ in Theorem 2, and assume that the distance from $g$ to $\mathcal{N}^2$ is less than $\varepsilon$, then after putting $x_i = rs_i$ and $y_i = rt_i$, we have either there are $s_i$ with $|s_i| \leq 1$ and

\[
|2s_1 + 1 + (s_1^2 - s_1)r| < 2\varepsilon/r,
\]

\[
|2s_2 + (s_2^2 - s_2)r| < 2\varepsilon/r,
\]

\[
|2s_1 - 2s_2 + 1 + (s_1^2 - s_2^2 + s_1 + 1)r| < 2\varepsilon/r;
\]

or there exist $t_i$ with $|t_i| \leq 1$ such that

\[
|t_1 - 1 + (t_1^2 - t_1)r| < 2\varepsilon/r,
\]

\[
|t_2 - 3 + (t_2^2 - t_2)r| < 2\varepsilon/r,
\]

\[
|t_1 - t_2 + 2 + (t_1^2 - t_1 + 1)t_1| < 2\varepsilon/r.
\]

We now show that if $\varepsilon = r^3/10$ and $r < .01$, then neither (9) nor (10) can be satisfied. For (10) this is immediate, since the second inequality in (10) implies that $|t_2 | < 2r + r^2/5$, contradicting $|t_1| \leq 1$. To show that (9) also fails, set

\[
A = 2s_1 + 1 + (s_1^2 - s_1)r, \quad B = 2s_2 + (s_2^2 - s_2)r,
\]

\[
C = 2s_1 - 2s_2 + 1 + (s_1^2 - s_2^2 + s_1 + 1)r,
\]

so that the inequalities in (10) become $|A| < r^2/5$, $|B| < r^2/5$, $|C| < r^2/5$. The first of these implies $|2s_1 + 1| \leq 2r + r^2/5 < 3r$, and hence that $s_1 = -\frac{1}{2} + ar$ where $|a| \leq \frac{3}{2}$. Substituting this into $A$, we have $|2a + \frac{2}{4}| \leq 3r + \frac{r}{5} + \frac{2}{4}r^2 < 4r$ and conclude that $s_1 = -\frac{1}{2} - \frac{3}{8}r + br$ where $|b| \leq 2$. In a similar way, $|B| < r^2/5$ implies that $s_2 = cr^2$ where $|c| \leq 1$. Finally, since $C = A - B + (2s_1 - 1 - s_2)r$, we find $|2s_1 - 1 - s_2| \leq \frac{3}{5}r$ and using the values obtained for $s_1$ and $s_2$, we finally obtain

\[
\left|\frac{3}{4} + (2b - c)r\right| \leq \frac{3}{5}
\]

which clearly cannot hold if $r < .01$.

To obtain the left side of (6), we merely observe that the special function $f_0(x, y) = \left(x + \frac{1}{2}y + 1\right)^2 - 1$ belongs to $\mathcal{N}^2$ and obeys $\|f_0 - g\|_{V_r} < r^2$. 

Having proved Theorem 2, we invoke Theorem 1 and obtain

**COROLLARY.** \( N^2 \text{ is a nowhere dense subset of } C[I^2]. \)

To show now that this is also true of the class \( N^k \) in the space \( C[I^k] \), we prove that one does not obtain a better nomographic approximation to the special function \( g \) given in Theorem 2 by using nomographic functions of \( k \) variables for any \( k > 2 \).

**THEOREM 3.** If \( g \) is any continuous function of \((x_1, x_2)\) then

\[
\inf_{f \in N^k} \|f - g\|_{I^k} = \inf_{f^* \in N^2} \|f^* - g\|_{I^2}. 
\]

**PROOF.** Let \( d \) be the distance in \( C[I^k] \) from \( g \) to \( N^k \). Given \( \delta > 0 \), choose \( f_0 \in N^k \) so that

\[
d_0 = \|f_0 - g\|_{I^k} < d + \delta. 
\]

Choose \( \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k) \) in \( I^k \) so that

\[
d_0 = \|f_0(\bar{x}) - g(\bar{x})\| = |h(\phi_1(\bar{x}_1) + \phi_2(\bar{x}_2) + \cdots + \phi_k(\bar{x}_k)) - g(\bar{x})|. 
\]

Define a function \( \psi \) on \( I \) by

\[
\psi(s) = \phi_2(s) + \phi_3(\bar{x}_3) + \cdots + \phi_k(\bar{x}_k),
\]

and then set \( f^*(x_1, x_2) = h(\phi_1(x_1) + \psi(x_2)) \). It is then clear that

\[
\|f^*(x_1, x_2) - g(x_1, x_2)\| = d_0 
\]

while for any \((x_1, x_2)\) in \( I^2 \),

\[
\|f^*(x_1, x_2) - g(x_1, x_2)\| = |f_0(x_1, x_2, \bar{x}_3, \ldots, \bar{x}_k) - g(x_1, x_2)| \leq \|f_0 - g\|_{I^k}. 
\]

Accordingly, \( \|f^* - g\|_{I^2} = \|f_0 - g\|_{I^k} < d + \delta \), holding for every \( \delta > 0 \). This proves one half of (11); the rest follows since \( N^2 \) is part of \( N^k \).

While five copies of \( N^2 \) are enough to give \( C[I^2] \) as their algebraic sum, it is known that four will not suffice. (See [5].) It would be of interest to know if the sum of four copies of \( N^2 \) is perhaps also nowhere dense in \( C[I^2] \). The argument we have used here will not work since the special function \( g(x, y) \) used in Theorem 2 in fact is already a member of \( N^2 + N^2 \). Indeed,

\[
g(x, y) = \exp(\log(x + 2) + \log(y + 3)) + (x^2 - x + y^2 - y - 6). 
\]

**REFERENCES**

1. V. I. Arnold, *On the representability of functions of two variables in the form \( \chi(\phi(x) + \psi(y)) \)*, Uspehi Mat. Nauk 12 (1957), 119–121. (Russian)


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