THE DEPENDENCE OF MASS UPON WIND

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ABSTRACT. As the winding number of unimodular function on \( \mathbb{R} \) increases must the mass of the measure from which it arises (via the Fourier transform) increase?

We work on the real numbers, \( \mathbb{R} \), \( \| \mu \|_M \) denotes the mass of measure \( \mu \) on \( \mathbb{R} \), and also we use \( \| \mu \|_B \) to denote the same \( \| \mu \|_M \) where \( \hat{\cdot} \) is the standard Fourier transform. We will also use \( \| f \|_B \) in the case that \( f \neq \hat{\mu} \) for any \( \mu \in M(T) \); in this case \( \| f \|_B \) denotes \( +\infty \). Let \( f \) be real valued, continuous and nonlinear. H. Helson, J. P. Kahane [K, p. 87] ask if \( \| e^{itf} \|_B \) must exhibit at least a specific growth rate with respect to \( r \). (The strong conjecture is \( \| e^{itf} \|_B > K \log r \)). Now in addition let \( f \) have a limit at \( +\infty \). We ask a more specific question. Does there exist a function \( \omega(r) \) with limit \( +\infty \) at \( +\infty \) and a constant \( K \) depending only on \( \text{Sup} f - \text{Inf} f \) such that \( \| e^{itf} \|_B > K \omega(r) \)? We motivate this question. Let \( g \) be a continuous unimodular function; write \( g = e^{itf} \) where \( f \) is continuous. If \( g \) has a finite winding number, then \( f \) has limits at \( \pm \infty \), say \( L \) and \( M \), and the winding number of \( g \) is \( (L - M)/2\pi \). Thus a yes answer to the question would show \( \| g \|_B > \omega \) (winding number of \( g \)) where \( \omega(r) \to +\infty \) as \( r \to +\infty \).

THEOREM. For every \( \delta > 0 \) there exists a function \( \omega_{\delta} \) such that \( \omega_{\delta}(r) \to +\infty \) as \( r \to +\infty \) that satisfies: If \( f \) is a real, nonconstant, continuous function on \( \mathbb{R} \) that has a limit at \( +\infty \) then

\[
\| e^{i\alpha_r - rf} \|_B > K \omega_{\delta}(r)
\]

where \( \alpha_r \in (1 - \delta, 1] \) and where \( K \) depends only on \( \text{Sup} f - \text{Inf} f \). Furthermore, there exists a function \( \omega \) such that \( \omega(r) \to +\infty \) as \( r \to +\infty \), that satisfies: if \( f \) is in addition monotone then

\[
\| e^{irf} \|_B > K \omega(r)
\]

where \( K \) depends only on \( \text{Sup} f - \text{Inf} f \).

The monotone case of the theorem has been obtained by A. M. Olevskii, [O].

THEOREM PROOF. We use \( \hat{\mu}(t) \) for \( \int t \, d\mu \). The proof uses the Cohen-Davenport proposition [G-McG, pp. 8-11] which we now state.

PROPOSITION. Suppose a system of \( Q^2 + 1 \) functions \( \{m_{00}\} \cup \{m_{ks} : k = 1, \ldots, Q^2; s = 1, \ldots, Q\} \) can be found that satisfy the following with respect to a measure \( \mu \). Let \( P_0 = \{m_{00}\} \). \( P_k = P_{k-1} \cup \{pm_{ks}\overline{m_{kt}} : p \in P_{k-1}, s < t\} \cup \{m_{ks}\} \).

(a) \( \|m_{ks}\|_\infty \leq 1 \).
(b) \( \hat{\mu}(m_{ks}) \geq 1 \).
(c) \( |\overline{\hat{\mu}(pm_{ks}\overline{m_{kt}})}| < e^{-Q} \) for \( p \in P_{k-1} \) and \( s < t \).

Then \( \| \mu \|_M > \sqrt{Q/4} \).

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We use these notations: \( \gamma_x(y) = e^{ixy} \). For any function \( g \) and any set \( S \) jump \( g \) on \( S = \sup_{x,y \in S} |g(x) - g(y)| \). Jump \( g = \text{Jump} \ g \) on \( \mathbb{R} \). Fix \( Q \in \mathbb{N} \). The smallness \( \epsilon = e^{-Q/3} \) will be used. Choose \( \lambda \) so that \( |x_{i+1}/x_i| > \lambda \) for \( i = 1, 2, \ldots \) insures \( \bigcap_{i \geq 2} \{ r: \gamma_{x_i}(r) \text{ is within } \epsilon \text{ of } -1; r \in \text{an interval } I \text{ where jump } \gamma_{x_i} \text{ on } I < \epsilon \} \neq \emptyset \). Fix \( \delta \) and choose \( \lambda_0 \) so that \( 2\pi/\lambda_0 < \delta \).

The proof idea is this. Provided only that jump \( f \) is sufficiently large compared to \( Q, \lambda_0 \) we will select two systems of real numbers \( \{x_{00}\} \cup \{y_{ks} : k = 1, \ldots, Q^2; s = 1, \ldots, Q\} \cup \{y_{00}\} \cup \{y_{ks} : k = 1, \ldots, Q^2; s = 1, \ldots, Q\} \) that satisfy

(i) \[ |f(x_{i2}) - f(y_{i2})| > \lambda_0. \]

(ii) For \( k = 1, 2, \ldots, Q^2; \quad s = 2, \ldots, Q \)

\[ |f(x_{k2}) - f(y_{k2})| > \lambda|f(x_{k(s-1)}) - f(y_{k(s-1)})| \]

and \( k = 2, 3, \ldots, Q^2 \)

\[ |f(x_{k2}) - f(y_{k2})| > \lambda|f(x_{(k-1)2}) - f(y_{(k-1)2})|. \]

(iii) Let \( F_0 = \{x_{00}, y_{00}\} \). \( F_k = F_{k-1} \cup \{z + z_{ks} - z_{kt} : z_{ks} = x_{ks} \text{ or } y_{ks}, z_{kt} = x_{kt} \text{ or } y_{kt}; s < t, z \in F_{k-1} \} \cup \{x_{ks}\} \cup \{y_{ks}\} \). \( \gamma(z + x_{ks} - x_{kt}) - f(x + x_{ks} - y_{kt})| < \epsilon \)

and

\[ |f(z + y_{ks} - x_{kt}) - f(z + y_{ks} - y_{kt})| < \epsilon \]

for \( z \in F_{k-1} \) and \( s < t \).

From the \( x, y \) systems we construct the \( m \)'s as follows. Select \( r (= 1) \) within \( 2\pi/\lambda_0 (\epsilon < \delta) \) of 1 so that \( e^{irf(x_{ks})} - f(y_{ks}) \) is within \( \epsilon \) of \(-1\) for all \( k \) and \( s = 2, \ldots, Q \). This is possible by (i), (ii). (This \( r \) plays the role of \( \alpha_r \) in the theorem; \( \omega_\delta \) is determined by the number needed to dilate \( f \) so that it has the required jump with respect to \( \lambda_0, Q \) needed to find \( \{x_{ks}, y_{ks}\} \).) Then let

\[ m_{ks} = \frac{1}{2}(e^{i\gamma x_{ks}}\gamma_{x_{ks}} + e^{i\gamma y_{ks}}\gamma_{y_{ks}}). \]

We check that this system satisfies (a), (b), (c) with respect to \( \hat{\mu} = e^{irf}. \) (a), (b) present no problems. To check (c) note that since \( p, m_{ks} \) have \( A \)-norm (= sum of the absolute values of their coefficients) \( \leq 1 \) we need only show: \( |\hat{\mu}(\gamma_{x_{ks}}\gamma_{y_{ks}}m_{ks})| < e^{-Q} \) and \( |\hat{\mu}(\gamma_{x_{ks}}\gamma_{y_{ks}}m_{kt})| < e^{-Q} \) for \( z \in F_{k-1} \) and \( t > s \) (so \( t \geq 2 \)). We do the first.

\[ \hat{\mu}(\gamma_{x_{ks}}\gamma_{y_{ks}}m_{kt}) = \frac{1}{2}(e^{-irf(x_{ks})}\hat{\mu}(z + z_{ks} - x_{kt}) + e^{-irf(y_{kt})}\hat{\mu}(z + z_{ks} - y_{kt}) + e^{-irf(y_{ks})}\hat{\mu}(z + z_{ks} - y_{kt}) + e^{-irf(x_{kt})}\hat{\mu}(z + z_{ks} - x_{kt})). \]

But \( e^{-irf(x_{ks})} \) is within \( e^{-Q/3} \) of \(-e^{-irf(y_{ks})} \) and \( r \leq 1 \) means \( e^{irf(z + z_{ks} - x_{kt})} \) is within \( e^{-Q/3} \) of \( e^{irf(z + z_{ks} - y_{kt})} \) (by (iii)) and so the summands almost cancel and hence (c).

We are thus left to select the \( x, y \) systems. The method is to overload the jumps of \( f \). Choose \( w \) so that jump (of \( f \) on \( [w, \infty) \) < \( \epsilon \). The existence of \( \lim_{z \to \infty} f(z) \) is needed here. Let \( x_{00} = y_{00} = w \). Let \( x_{11} = y_{11} = x_{00} \). Choose \( w \) so that jump \( [w, y_{11}] = \lambda_0 \). Let \( x_{12} > y_{12} \) be jump points. Suppose \( x_{1s}, y_{1s} \) for \( s < t \) are known with \( x_{11} \geq y_{11} \geq \cdots \geq x_{1(t-1)} \geq y_{1(t-1)} \). Choose \( w \) so that \( [w, y_{1(t-1)}] \) has jump \( \lambda^{(t-1)}\lambda_0 \) and let \( x_{st} > y_{st} \) be jump points. Note that the jump in \( f \), denoted \( J_1 \), needed to form \( F_1 \) is at most \( \epsilon + \lambda_0 + \lambda_0\lambda + \cdots + \lambda_0\lambda^{(Q-1)} \).
The rest of the argument is to show that knowledge of $J_{k-1}$ gives knowledge of $J_k$. Thus the required jump $J_{Q^2}$ can be computed recursively as a function of $\lambda_0, Q$.

Suppose $F_{k-1}$ is known. Card $F_{k-1}$ depends only on $Q, \lambda_0$; we call it $L$. Let $x_{k_1} = y_{k_1} = x_0$. Let $[w, y_{k_1}]$ have jump, $j_2 = \lambda \cdot J_{k-1}(J_{k-1}/\epsilon)^L$. $x_{k_2}, y_{k_2}$ will be selected from $[w, y_{k_1}]$, and hence jump on $x_{k_2}, y_{k_2} < j_2$. List $F_{k-1} = z_1, \ldots, z_L$. Now we overload jumps. From $[w, y_{k_1}]$ select a subset $S_1$ with (Lebesgue) measure of $f(S_1) > J_{k-1}\lambda(J_{k-1}/\epsilon)^{L-1}$ such that jump on $z_1 + x_{k_1} - S_1 < \epsilon$; this is possible since $[z_1, +\infty)$ can be partitioned into $J_{k-1}/\epsilon \epsilon$-jumps. Select from $S_1$ a subset $S_2$ with measure of $f(S_2) > J_{k-1}\lambda(J_{k-1}/\epsilon)^{L-2}$ and jump on $x_2 + x_{k_1} - S_2 < \epsilon$. Continuing in this manner we obtain a set $S$ with measure $f(S) > J_{k-1}\lambda$ ($> |f(x(k-1)Q) - f(y(k-1)Q)|$) and jump on $z_i + x_{k_1} - S (= z_i + y_{k_1} - S) < \epsilon$ for all $i$. Choose $x_{k_2} \geq y_{k_2}$ points of $S$ that exhibit at least a $J_{k-1}$ jump. Now suppose $x_{k_1} = y_{k_1} \geq \cdots \geq x_{k(t-1)} \geq y_{k(t-1)}$ are known, and suppose $x_{k(t-1)}, y_{k(t-1)}$ were selected from $[w, y_{k(t-2)}]$ having jump $j_{t-1}$. Choose $w$ so that $[w, y_{k(t-1)}]$ has jump $j_t = j_{t-1}\lambda(J_{k-1}/\epsilon)^{L-2t}$. From $[w, y_{k(t-1)}]$ we select a set $S$ with measure $f(S) > j_t \lambda$ and such that jump $z_i + x_{k_2} - S < \epsilon$ and jump $z_i + y_{k_2} - S < \epsilon$ for all $i$ and $s < t$. Let $x_{kt} \geq y_{kt}$ be points that exhibit at least a $j_{t-1} \cdot \lambda$ jump. Thus, $J_k = \epsilon + j_2 + \cdots + j_{Q^2}$ works and the main theorem is proved.

Now if $f$ is in addition monotone the sets $S$ of the previous argument can be chosen to be intervals and, hence, we can choose points in $S$ having jump exactly $\pi$.

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References


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