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THE DEPENDENCE OF MASS UPON WIND

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ABSTRACT. As the winding number of unimodular function on $\mathbf{R}$ increases must the mass of the measure from which it arises (via the Fourier transform) increase?

We work on the real numbers, $\mathbf{R}$, $\|\mu\|_M$ denotes the mass of measure μ on $\mathbf{R}$, and also we use $\|\hat{\mu}\|_B$ to denote the same $\|\mu\|_M$ where $\hat{}$ is the standard Fourier transform. We will also use $\|f\|_B$ in the case that $f \neq \hat{\mu}$ for any $\mu \in M(T)$; in this case $\|f\|_B$ denotes $+\infty$. Let f be real valued, continuous and nonlinear. H. Helson, J. P. Kahane [**K**, p. 87] ask if $\|e^{irf}\|_B$ must exhibit at least a specific growth rate with respect to r. (The strong conjecture is $\|e^{irf}\|_B > K \log r$). Now in addition let f have a limit at $+\infty$. We ask a more specific question. Does there exist a function $\omega(r)$ with limit $+\infty$ at $+\infty$ and *a constant* K *depending only on* $\operatorname{Sup} f - \operatorname{Inf} f$ such that $\|e^{irf}\|_B > K\omega(r)$? We motivate this question. Let g be a continuous unimodular function; write $g = e^{if}$ where f is continuous. If g has a finite winding number, then f has limits at $\pm\infty$, say L and M, and the winding number of g is $(L - M)/2\pi$. Thus a yes answer to the qustion would show $\|g\|_B > \omega$ (winding number of g) where $\omega(r) \to \infty$ as $r \to \infty$.

THEOREM. *For every $\delta > 0$ there exists a function ω_δ such that $\omega_\delta(r) \to +\infty$ as $r \to +\infty$ that satisfies: If f is a real, nonconstant, continuous function on* $\mathbf{R}$ *that has a limit at $+\infty$. then*

$$\|e^{i \cdot \alpha_r \cdot rf}\|_B > K\omega_\delta(r)$$

where $\alpha_r \in (1 - \delta, 1]$ and where K depends only on $\operatorname{Sup} f - \operatorname{Inf} f$. *Furthermore, there exists a function ω such that $\omega(r) \to +\infty$ as $r \to +\infty$, that satisfies: if f is in addition monotone then*

$$\|e^{irf}\|_B > K\omega(r)$$

where K depends only on $\operatorname{Sup} f - \operatorname{Inf} f$.

The monotone case of the theorem has been obtained by A. M. Olevskii, [**0**].

THEOREM PROOF. We use $\hat{\mu}(t)$ for $\int \bar{t}\, d\mu$. The proof uses the Cohen-Davenport proposition [**G-McG**, pp. 8-11] which we now state.

PROPOSITION. *Suppose a system of $Q^3 + 1$ functions $\{m_{00}\} \cup \{m_{ks} : k = 1, \ldots, Q^2; s = 1, \ldots, Q\}$ can be found that satisfy the following with respect to a measure μ: Let $P_0 = \{m_{00}\}$. $P_k = P_{k-1} \cup \{pm_{ks}\overline{m_{kt}} : p \in P_{k-1},\, s < t\} \cup \{m_{ks}\}$.*

(a) $\|m_{ks}\|_\infty \leq 1$.

(b) $\hat{\mu}(m_{ks}) \geq 1$.

(c) $|\hat{\mu}(pm_{ks}\overline{m_{kt}})| < e^{-Q}$ *for $p \in P_{k-1}$ and $s < t$.*

Then $\|\mu\|_M > \sqrt{Q}/4$.

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We use these notations: $\gamma_x(y) = e^{ixy}$. For any function g and any set S jump g on $S = \sup_{x,y\in S} |g(x) - g(y)|$. Jump $g =$ Jump g on $\mathbf{R}$. Fix $Q \in \mathbf{N}$. The smallness $\epsilon = e^{-Q}/3$ will be used. Choose λ so that $|x_{i+1}/x_i| > \lambda$ for $i = 1, 2, \ldots$ insures $\bigcap_{i\geq 2}\{r\colon \gamma_{x_i}(r)$ is within ϵ of -1; $r \in$ an interval I where jump γ_{x_1} on $I < \epsilon\} \neq \emptyset$. Fix δ and choose λ_0 so that $2\pi/\lambda_0 < \delta$.

The proof idea is this. Provided only that jump f is sufficiently large compared to Q, λ_0 we will select two systems of real numbers $\{x_{00}\} \cup \{x_{ks} : k = 1, \ldots, Q^2; s = 1, \ldots, Q\}$; $\{y_{00}\} \cup \{y_{ks} : k = 1, \ldots, Q^2;\ s = 1, \ldots, Q\}$ that satisfy

(i) $|f(x_{12}) - f(y_{12})| > \lambda_0$.

(ii) For $k = 1, 2, \ldots, Q^2$; $\quad s = 2, \ldots, Q$

$$|f(x_{ks}) - f(y_{ks})| > \lambda |f(x_{k(s-1)}) - f(y_{k(s-1)})|$$

and for $k = 2, 3, \ldots, Q^2$

$$|f(x_{k2}) - f(y_{k2})| > \lambda |f(x_{(k-1)Q}) - f(y_{(k-1)Q})|.$$

(iii) Let $F_0 = \{x_{00}, y_{00}\}$. $F_k = F_{k-1} \cup \{z + z_{ks} - z_{kt}\colon z_{ks} = x_{ks}$ or $y_{ks}, z_{kt} = x_{kt}$ or $y_{kt}\colon s < t, z \in F_{k-1}\} \cup \{x_{ks}\} \cup \{y_{ks}\}$.

$$|f(z + x_{ks} - x_{kt}) - f(z + x_{ks} - y_{kt})| < \epsilon$$

and

$$|f(z + y_{ks} - x_{kt}) - f(z + y_{ks} - y_{kt})| < \epsilon$$

for $z \in F_{k-1}$ and $s < t$.

From the x, y systems we construct the m's as follows. Select $r\,(\leq 1)$ within $2\pi/\lambda_0\,(< \delta)$ of 1 so that $e^{ir(f(x_{ks}) - f(y_{ks}))}$ is within ϵ of -1 for all k and $s = 2, \ldots, Q$. This is possible by (i), (ii). (This r plays the role of α_r in the theorem; ω_δ is determined by the number needed to dilate f so that it has the required jump with respect to λ_0, Q needed to find $\{x_{ks}, y_{ks}\}$.) Then let

$$m_{ks} = \tfrac{1}{2}\big(e^{irf(x_{ks})}\gamma_{x_{ks}} + e^{irf(y_{ks})}\gamma_{y_{ks}}\big).$$

We check that this system satisfies (a), (b), (c) with respect to $\hat{\mu} = e^{irf}$. (a), (b) present no problems. To check (c) note that since p, m_{ks} have A-norm ($=$ sum of the absolute values of their coefficients) ≤ 1 we need only show: $|\hat{\mu}(\gamma_z\gamma_{x_{ks}}\overline{m}_{kt})| < e^{-Q}$ and $|\hat{\mu}(\gamma_z\gamma_{y_{ks}}\overline{m}_{kt})| < e^{-Q}$ for $z \in F_{k-1}$ and $t > s$ (so $t \geq 2$). We do the first.

$$\begin{aligned}\hat{\mu}(\gamma_z\gamma_{x_{ks}}\overline{m}_{kt}) &= \tfrac{1}{2}\big(e^{-irf(x_{kt})}\hat{\mu}(z + x_{ks} - x_{kt}) + e^{-irf(y_{kt})}\hat{\mu}(z + x_{ks} - y_{kt})\big)\\ &= \tfrac{1}{2}\big(e^{-irf(x_{kt})}e^{irf(z+x_{ks}-x_{kt})} + e^{-irf(y_{kt})}e^{irf(z+x_{ks}-y_{kt})}\big).\end{aligned}$$

But $e^{-irf(x_{kt})}$ is within $e^{-Q}/3$ of $-e^{irf(y_{kt})}$ and $r \leq 1$ means $e^{irf(z+x_{ks}-x_{kt})}$ is within $e^{-Q}/3$ of $e^{irf(z+x_{ks}-y_{kt})}$ (by (iii)) and so the summands almost cancel and hence (c).

We are thus left to select the x, y systems. The method is to overload the jumps of f. Choose w so that jump (of f) on $[w, \infty) < \epsilon$. The existence of $\lim_{x\to\infty} f(x)$ is needed here. Let $x_{00} = y_{00} = w$. Let $x_{11} = y_{11} = x_{00}$. Choose w so that jump $[w, y_{11}] = \lambda_0$. Let $x_{12} > y_{12}$ be jump points. Suppose x_{1s}, y_{1s} for $s < t$ are known with $x_{11} \geq y_{11} \geq \cdots \geq x_{1(t-1)} \geq y_{1(t-1)}$. Choose w so that $[w, y_{1(t-1)}]$ has jump $\lambda^{(t-1)}\lambda_0$ and let $x_{1t} > y_{1t}$ be jump points. Note that the jump in f, denoted J_1, needed to form F_1 is at most $\epsilon + \lambda_0 + \lambda_0\lambda + \cdots + \lambda_0\lambda^{(Q-1)}$.

The rest of the argument is to show that knowledge of J_{k-1} gives knowledge of J_k. Thus the required jump J_{Q^2} can be computed recursively as a function of λ_0, Q.

Suppose F_{k-1} is known. Card F_{k-1} depends only on Q, λ_0; we call it L. Let $x_{k1} = y_{k1} = x_{00}$. Let $[w, y_{k1}]$ have jump, $j_2, = \lambda \cdot J_{k-1}(J_{k-1}/\epsilon)^L$. x_{k2}, y_{k2} will be selected from $[w, y_{k1}]$, and hence jump on $x_{k2}, y_{k2} < j_2$. List $F_{k-1} = z_1, \ldots, z_L$. Now we overload jumps. From $[w, y_{k1}]$ select a subset S_1 with (Lebesgue) measure of $f(S_1) > J_{k-1}\lambda(J_{k-1}/\epsilon)^{L-1}$ such that jump on $z_1 + x_{k1} - S_1 < \epsilon$; this is possible since $[z_1, +\infty)$ can be partitioned into J_{k-1}/ϵ ϵ-jumps. Select from S_1 a subset S_2 with measure of $f(S_2) > J_{k-1}\lambda(J_{k-1}/\epsilon)^{L-2}$ and jump on $z_2 + x_{k1} - S_2 < \epsilon$. Continuing in this manner we obtain a set S with measure $f(S) > J_{k-1}\lambda$ $(> |f(x_{(k-1)Q}) - f(y_{(k-1)Q})|)$ and jump on $z_i + x_{k1} - S$ $(= z_i + y_{k1} - S) < \epsilon$ for all i. Choose $x_{k2} \geq y_{k2}$ points of S that exhibit at least a $J_{k-1}\lambda$ jump. Now suppose $x_{k1} = y_{k1} \geq \cdots \geq x_{k(t-1)} \geq y_{k(t-1)}$ are known, and suppose $x_{k(t-1)}, y_{k(t-1)}$ were selected from $[w, y_{k(t-2)}]$ having jump j_{t-1}. Choose w so that $[w, y_{k(t-1)}]$ has jump $j_t = j_{t-1}\lambda(J_{k-1}/\epsilon)^{L \cdot 2t}$. From $[w, y_{k(t-1)}]$ we select a set S with measure $f(S) > j_{t-1}\lambda$ and such that jump $z_i + x_{ks} - S < \epsilon$ and jump $z_i + y_{ks} - S < \epsilon$ for all i and $s < t$. Let $x_{kt} \geq y_{kt}$ be points that exhibit at least a $j_{t-1} \cdot \lambda$ jump. Thus, $J_k = \epsilon + j_2 + \cdots + j_{Q^2}$ works and the main theorem is proved.

Now if f is in addition monotone the sets S of the previous argument can be chosen to be *intervals* and, hence, we can choose points in S having jump exactly π.

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