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NORM ATTAINING OPERATORS AND NORMING FUNCTIONALS

RUSSELL G. BILYEU AND PAUL W. LEWIS

ABSTRACT. The question of whether a countably additive measure with values in a Banach space attains the diameter of its range was unresolved. In this paper an example is given of a countably additive vector measure, taking values in a $C(K)$ space, for which the diameter of the range is not attained. A property stronger than the attainment of the diameter, but which is possessed by many measures taking values in L-spaces, is shown to fail for infinite-dimensional measures into a space having smooth dual. As an application of the concept of norming functional (the existence of which is equivalent to the attainment of diameter), a characterization is given of the countably additive measures into space having smooth dual.

I. Introduction. Suppose that (Ω, Σ) is a measurable space, X is a Banach space, and $\mu : \Sigma \to X$ is a countably additive vector measure. The space $\mathrm{ca}(\Sigma, X)$ of all such measures is a Banach space under the semivariation norm: $\|\mu\| = \sup\{|x^*\mu|(\Omega): x^* \in B_{X^*}\}$, where B_{X^*} denotes the closed unit ball of X^* and $|x^*\mu|$ denotes the total variation of the scalar measure $x^*\mu$. It is not difficult to see that $\|\mu\| = \mathrm{diameter}(\mu(\Sigma))$, i.e. $\|\mu\| = \sup\{\|\mu(A) - \mu(B)\colon A, B \in \Sigma\}$. In this note we are concerned with the problem of whether this diameter is actually attained and resultant implications for the range of the vector measure. This problem is equivalent to the existence of a *norming functional* for μ. An element $x^* \in B_{X^*}$ is called a norming functional for μ if $|x^*\mu|(\Omega) = \|\mu\|$. Further, if $U(\Sigma)$ denotes the uniform closure of the simple functions defined over Σ, and μ corresponds to the weakly compact operator $T\colon U(\Sigma) \to X$ in the natural manner (cf. Dinculeanu [**7**], Diestel and Uhl [**6**, Chapter VI] or Brooks and Lewis [**4**]), then it is immediate that μ attains its diameter iff there is an element $x^* \in B_{X^*}$ such that $\|T^*\| = \|T^*(x^*)\|$ (T^* is norm attaining [**10**]). The remainder of the introduction provides some background as well as the setting for the specific versions of this problem that we shall address.

We begin with a much used example of a countably additive vector measure with finite variation which fails to have a Radon-Nikodym derivative, yet has a norming functional in a very strong way. Let $\mathcal{B}$ denote the Borel subsets of $[0,1]$, and define $\mu : \mathcal{B} \to L^1[0,1]$ by $\mu(A) = \chi_A$. The range of μ is not relatively compact (and consequently must be infinite dimensional), and the function $g \equiv 1$ is a norming functional. In fact, if $B \in \mathcal{B}$ and $\mu_B : \mathcal{B} \to L^1[0,1]$ is defined by $\mu_B(A) = \mu(A \cap B)$, then g is a norming functional for μ_B also. Such a functional will be termed a hereditary norming functional. In Theorem 2.2 we show that an analogous example (i.e. with infinite-dimensional range) is not possible if X is any Banach space with

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a smooth dual (such as $L^2[0,1]$). Our argument uses ideas from [**3**]. We note that in §2 of [**3**], Bilyeu and Lewis show that if X^* is smooth and μ is a nonzero element of $\mathrm{ca}(\Sigma, X)$ with norming functional x^*, then x^* satisfies the conclusion of Rybakov's Theorem [**11, 6**]: $y^*\mu \ll x^*\mu$ for all $y^* \in X^*$. Consequently, x^* strongly exposes $\mu(\Sigma)$ [**1, 6**]. Finite-dimensional examples show that the smoothness condition cannot be dropped from X^*. We also remark that if μ has relatively compact range, then μ attains its diameter. This follows from Lemma 2.7 of [**3**], as well as from the fact that the adjoint of a compact operator is always norm attaining [**8**, Chapter VI].

In view of the recent paper by Professors Johnson and Wolfe concerning norm attaining operators [**10**], the question about the existence of norming functionals seems to be of particular interest when X is the Banach space $C(K)$ (sup norm) of all continuous functions on a compact Hausdorff space K. In Theorem 1 of [**10**], Johnson and Wolfe show that the norm attaining bounded linear transformations (= operators) from $C(S)$ to $C(K)$ are dense in the space of all operators from $C(S)$ to $C(K)$. (S is a compact Hausdorff space). Since the adjoint of a norm attaining operator is certainly norm attaining (by the Hahn-Banach Theorem) and $U(\Sigma)$ is isometrically a $C(S)$ space, it follows immediately from Theorem 1 of [**10**] that there is an abundance of elements in $\mathrm{ca}(\Sigma, C(K))$ which do not necessarily have relatively compact range and yet possess norming functionals. In the next section we use techniques from [**10**] to construct a countably additive measure μ with values in the space c_0 so that μ fails to attain its diameter.

II. Norming functionals. In this section we produce the example just mentioned, and we give a characterization of vector measures which have hereditary norming functionals.

2.1. EXAMPLE. Let K be the subset of $[0,1]$ consisting of $\{\frac{1}{n}: n = 1,2,\dots\} \cup \{0\}$; endow K with its relative topology. Then $C(K)$ is the space c of all convergent sequences. Define $\tau: K \to l^2$ as follows:

$\tau(\frac{1}{n}) = (1 - \frac{1}{n})e_n$, where e_n is the nth canonical basis element, $n = 1,2,\dots$,
$\tau(0) = 0$.

Then $\|\tau(k)\| < 1$ for all $k \in K$, and $\sup\|\tau(k)\| = 1$. Furthermore, τ is a continuous function when l^2 is given its weak topology. By Theorem 1 of Dunford and Schwartz [**8**, p. 490], there is a weakly compact operator $T: l^2 \to c$ so that $\|T\| = \sup_K \|\tau(k)\| = 1$ and $T^*(\delta_k) = \tau(k)$ for each $k \in K$ (δ_k denotes the point mass measure concentrated at k). Since each element of B_{c^*} may be written as an infinite linear combination $\sum_K \lambda_k\delta_k$, where $\sum|\lambda_k| \le 1$, it follows that T^* (and consequently T) does not attain its norm.

Next, let $\mathcal{A}$ be the σ-algebra generated by the dyadic intervals $[0,1]$, $[0,\frac{1}{2}]$, $[\frac{1}{2},1]$, $[0,\frac{1}{4}]$, $[\frac{1}{4},\frac{2}{4}]$, $[\frac{2}{4},\frac{3}{4}]$, $[\frac{3}{4},1]$, etc. Let N be the natural map from $U(\mathcal{A})$ into $L^2[0,1]$. Then $\|N\| = 1$, and the Walsh functions $\mathcal{W}$ belong to the image of the unit ball in $U(\mathcal{A})$. Let I be any linear isometry of $L^2[0,1]$ onto l^2 which maps $\mathcal{W}$ onto $\{e_i: i = 1,2,\dots\}$.

Now we consider the weakly compact operator $TIN: U(\mathcal{A}) \to c$. Since $\|T\| = \|I\| = \|N\| = \|TIN\| = 1$ and T^* does not attain its norm, it follows that $(TIN)^* = N^*I^*T^*$ does not attain its norm.

Consequently, from remarks in the introduction, it follows that the member of $\mathrm{ca}(\mathcal{A}, c)$ which represents TIN does not have a norming functional. This measure μ may be described explicitly as follows. Let (w_n) be an enumeration of $\mathcal{W}$ so that

$I(w_n) = e_n$, $n = 1, 2, \ldots$. Then

$$\mu(A) = \left(\left(1 - \frac{1}{n}\right)\int_A w_n \, d\lambda\right)_{n=1}^{\infty},$$

where $A \in \mathcal{A}$ and $\lambda =$ Lebesgue measure.

We recall that an arbitrary nonzero element x in a Banach space X is said to be a smooth point if the Gateaux derivative $D(x, y)$ of the norm at x in the direction y exists for each $y \in X$; the space X is said to be smooth if each point on the surface of B_X is a smooth point.

2.2 THEOREM. *Suppose that X^* is smooth and that μ is a nonzero element of* $\mathrm{ca}(\Sigma, X)$. *If $y^* \in X^*$ and y^* annihilates the range of μ, then $D(x^*, y^*) = 0$ whenever x^* is a norming functional for μ. Conversely, if x^* is a hereditary norming functional for μ, then $|y^*\mu|(\Omega) = 0$ whenever $D(x^*, y^*) = 0$. Thus μ has a hereditary norming functional iff $\mu(\Sigma)$ is one dimensional.*

PROOF. Suppose that x^* is a norming functional for μ, and let $y^* \in X^*$ so that $|y^*\mu|(\Omega) = 0$, i.e., y^* annihilates the range of μ. Since X^* is smooth, $D(x^*, \cdot)$ defines a continuous linear functional on X^*. Let α be a real number, and let m^* be an element in the kernel of $D(x^*, \cdot)$ so that $y^* = m^* - \alpha x^*$. Since $|y^*\mu|(\Omega) = 0$, it follows that $m^*\mu(A) = \alpha x^*\mu(A)$ for all $A \in \Sigma$. Furthermore, from Theorem 2.3 of [**3**], we see that $D(x^*\mu, m^*\mu) = 0$ (since $\|x^*\mu\| = \|\mu\|$ and $D(x^*, m^*) = 0$). From Theorem 3.1 of [**3**],

$$D(x^*\mu, m^*\mu) = \int_\Omega \mathrm{sgn}(x^*\mu) m^*\mu,$$

where this integral is the refinement integral studied by Appling [**2**]. Thus $D(x^*\mu, m^*\mu) = m^*\mu(A) - m^*\mu(B)$, where (A, B) form a Hahn decomposition of Ω for $x^*\mu$. Therefore $m^*\mu(A) = m^*\mu(B)$. Without loss of generality, we assume that $\alpha > 0$. But then $m^*\mu(A) \le 0$ and $m^*\mu(B) \ge 0$. Consequently, $m^*\mu(A) = m^*\mu(B) = 0 = \alpha x^*\mu(A) = \alpha x^*\mu(B)$, and μ must be trivial. But this is a contradiction. Thus $\alpha = 0$, and the first part of the theorem follows.

Now we assume that x^* is a hereditary norming functional, $\|y^*\| = 1$, and $D(x^*, y^*) = 0$. As above, we let (A, B) be a Hahn decomposition of Ω for $x^*\mu$, where A is a positive set. Again, from 2.3 and 3.1 of [**3**], $D(x^*\mu, y^*\mu) = y^*\mu(A) - y^*\mu(B) = 0$.

Now suppose that $\|y^*\mu\| \neq 0$, and let $C \in \Sigma$ so that $y^*\mu(C) > 0$. Without loss of generality, suppose that $y^*\mu(C \cap A) > 0$. But x^* is a norming functional for $\mu_{C\cap A}$, and $D(x^*\mu_{C\cap A},\ y^*\mu_{C\cap A}) = 0$ (Theorem 2.3 of [**3**]). Consequently,

$$D(x^*\mu_{C\cap A}, y^*\mu_{C\cap A}) = 0 = \int_\Omega \mathrm{sgn}(x^*\mu_{C\cap A}) y^*\mu_{C\cap A} = y^*\mu(C \cap A),$$

and we have reached a contradiction. Thus $\|y^*\mu\| = 0$.

Continuing our assumption that x^* is hereditarily norming, we now assert that $D(x^*, \cdot) \in X$. For suppose that (A, B) is a Hahn decomposition for $x^*\mu$. Then

$$\|\mu\| = \|x^*\mu\| = x^*\mu(A) - x^*\mu(B) \le \|\mu(A) - \mu(B)\| \le \|\mu\|.$$

Thus if we put $x_0 = (\mu(A) - \mu(B))/\|\mu\|$, then $x_0 \in B_X$ and $x^*(x_0) = 1$. But, as a result of the smoothness of X^* [**5**, Theorem 2.1], there is a unique element in $B_{X^{**}}$ (namely $D(x^*, \cdot)$) which takes the value 1 at x^*. Therefore $D(x^*, \cdot) = x_0$.

Now suppose that $x \in \mu(\Sigma)$ and $x \notin \operatorname{span}(x_0)$. Let $y^* \in X^*$ so that $y^*(x_0) = 0$, and $y^*(x) \neq 0$. Then $y^*(x_0) = D(x^*, y^*) = 0$, and thus $|y^*\mu|(\Omega) = 0$ by the first part of the theorem. But this is a contradiction since $y^*(x) \neq 0$, and it follows that $\mu(\Sigma)$ is one dimensional. Since it is clear that one-dimensional measures have hereditarily norming functionals, the theorem follows.

REMARK. The proof shows that the assumption that μ has a smooth hereditary norming functional (rather than the assumption that the entire space X^* is smooth) implies that $\mu(\Sigma)$ is one dimensional.

III. Orthogonality. In this section we use the idea of norming functionals in combination with James' nonlinear orthogonality [9] to give a geometric characterization of the countably additive members of $\mathrm{sa}(\Sigma, X)$, where $\mathrm{sa}(\Sigma, X)$ denotes the strongly additive members of the space $\mathrm{ba}(\Sigma, X)$ of all bounded finitely additive X-valued measures defined over Σ. The reader may consult Chapter I of Diestel and Uhl [6] for a discussion of many of the properties of $\mathrm{ba}(\Sigma, X)$ and $\mathrm{sa}(\Sigma, X)$. We continue to let $||\mu||$ denote the semivariation norm of a member of $\mathrm{ba}(\Sigma, X)$.

Following James, if x and y are arbitrary elements of a Banach space X, then we say that x is orthogonal to y (and write $x \perp y$) if $\inf\{||x + \alpha y|| : \alpha \in \mathbf{R}\} = ||x||$. If S is a nonempty subset of X, then we write $x \perp S$ if $x \perp y$ for all $y \in S$.

In the following theorem we denote the purely finitely additive members of $\mathrm{sa}(\Sigma, X)$ by $\mathrm{pfa}(\Sigma, X)$ [6, Chapter I], and we let $\sigma[(A_i)]$ denote the σ-ring generated by a sequence (A_i) from Σ. Additionally, if $\nu \in sa(\Sigma, X)$, then $\nu|_{(\sigma[(A_i)]}$ will denote the restriction of ν to $\sigma[(A_i)]$, and $\wedge$ will be used to denote the lattice min operation.

3.1 THEOREM. *If X^* is smooth and $\nu \in \mathrm{sa}(\Sigma, X)$, then ν is countably additive iff $\nu|_{[(A_i)]} \perp \mathrm{pfa}(\sigma[(A_i)], X)$ for each disjoint sequence (A_i) from Σ.*

PROOF. Suppose that $\nu \in \mathrm{ca}(\Sigma, X)$, (A_i) is a disjoint sequence from Σ, and $m \in \mathrm{pfa}([(A_i)], X)$. Denote the restriction of ν to $\sigma[(A_i)]$ by ξ, and let $t \in \mathbf{R}$. Since $||x^*(tm + \xi)|| = ||x^*tm|| + ||x^*\xi||$ for each $x^* \in X^*$, it follows that

$$||\xi|| = \sup\{||x^*\xi \colon x^* \in B_{X^*}\} \leq \sup\{||x^*(tm + \xi) \colon x^* \in B_{X^*}\} = ||tm + \xi||.$$

Consequently, $\xi \perp \mathrm{pfa}(\sigma[(A_i)], X)$.

Conversely, suppose that (A_i) is a disjoint sequence from Σ and $\nu|_{\sigma[(A_i)]} \perp \mathrm{pfa}(\sigma[(A_i)], X)$. By the Hewitt-Yosida decomposition theorem [6], we may (and shall) decompose $\nu|_{\sigma[(A_i)]} = \xi$ into a countably additive part ξ_c and a purely finitely additive part ξ_f, i.e., $\xi = \xi_c + \xi_f$.

We assert that there is an $x^* \in B_{x^*}$ so that

$$||x^*\xi_c|| = |x^*\xi_c|\left(\bigcup_{i=1}^{\infty} A_i\right) = ||\xi_c||,$$

i.e., ξ_c has a norming functional. We begin by considering the function ψ defined on the compact metric space $M = \{-1, 1\}^N$ by

$$\psi(g) = \sum_{i \in P_g} \xi_c(A_i) - \sum_{i \in N_g} \xi_c(A_i),$$

where $P_g = \{i: g(i) = 1\}$ and $N_g = \{i: g(i) = -1\}$. Now suppose that $g_k \to g$ in M. Consequently $\chi_{P_k} - \chi_{N_k} \to \chi_{P_g} - \chi_{N_g}$ pointwise on $M(P_k = P_{g_k}, N_k = N_{g_k})$. But then identifying i and A_i, it follows that

$$\psi(g_k) = \int (\chi_{P_k} - \chi_{N_k})\, d\xi_c \to \int (\chi_{P_g} - \chi_{N_g})\, d\xi_c = \psi(g)$$

by the bounded convergence theorem [**6**, p. 56]. Thus ψ is continuous, and $\psi(M)$ is compact. Hence there is a subset S of N so that

$$\left\| \xi_c\left(\bigcup_{i\in S} A_i\right) - \xi_c\left(\bigcup_{i\notin S} A_i\right) \right\| = \sup\{\|\psi(g)\| : g \in M\} = \|\xi_c\|.$$

Let $x_0^* \in SX^*$ so that

$$x_0^*\left(\xi_c\left(\bigcup_{i\in S} A_i\right) - \xi_c\left(\bigcup_{i\notin S} A_i\right)\right) = \left\| \xi_c\left(\bigcup_{i\in S} A_i\right) - \xi_c\left(\bigcup_{i\notin S} A_i\right) \right\|;$$

then $\|x_0^*\xi_c\| = \|\xi_c\|$.

Next we observe that $\|\xi\| = \|\xi_c\|$ since $\xi \perp \mathrm{pfa}(\sigma[(A_i)], X)$. Thus $\|x_0^*\xi\| = \|\xi\|$, and $x_0^*\xi$ is a control measure for ξ by Corollary 2.4 of [**3**], i.e. $y^*\xi \ll |x_0^*\xi|$ for all $y^* \in X^*$. Then combining the facts that

$$|u^*\xi_c| \wedge |v^*\xi_f| = 0 \quad \text{for } u^*, v^* \in X^*$$

and $x_0^*\xi$ is a control measure for ξ, it follows that $y^*\xi_c \ll |x_0^*\xi_c|$ and $y^*\xi_f \ll |x_0^*\xi_f|$ for $y^* \in X^*$. However, $x_0^*\xi_f = 0$ since

$$\|\xi_c\| = \|\xi\| = \|x_0^*\xi\| = \|x_0^*\xi_c\| + \|x_0^*\xi_f\| = \|\xi_c\| + \|x_0^*\xi_f\|.$$

Thus $y^*\xi_f = 0$ for all y^*, and the converse follows when one observes that ν is countably additive iff $\nu|_{\sigma[(A_i)]}$ is countably additive for each disjoint (A_i) from Σ.

We conclude with a remark and a question related to material in this paper.

1. If the values of the measure μ constructed in Example 2.1 are viewed as elements of l^2, then μ has its values in a reflexive space and still does not have a norming functional.

2. Can the compactness assumption be removed from Theorem 2.7 of [**3**]? This assumption guaranteed the existence of a norming functional and facilitated the argument in [**3**].

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DEPARTMENT OF MATHEMATICS, NORTH TEXAS STATE UNIVERSITY, DENTON, TEXAS 76203