PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 2, June 1982

A TOPOLOGICAL VERSION OF SLUTSKY'S THEOREM

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ABSTRACT. For weak convergence of probability measures on a product of two topological spaces the convergence of the marginals is certainly necessary. If however the marginals on one of the factor spaces converge to a one-point measure, the condition becomes sufficient, too. This generalizes a well-known result of Slutsky.

One of the most frequently applied theorems in Mathematical Statistics is the so-called "Slutsky's theorem". Roughly stated this theorem says that if a sequence of random variables converges in distribution to a certain limit law, then so does a slightly disturbed sequence. More precisely: let $X_1, X_2, \ldots$ be a sequence of real valued random variables converging in distribution to μ. Let $Y_1, Y_2, \ldots$ be another sequence of random variables defined on the same probability space, but not necessarily independent of the X_j. Then if Y_n converges in probability to zero, the "disturbed" sequence $X_n + Y_n$ converges in distribution to μ. Similarly, if Y_n converges to one then μ is the limit law of $X_n Y_n$, cf. [3].

More recently, in connection with infinite dimensional random variables (resp. stochastic processes), a generalisation of Slutsky's theorem was obtained for random variables with values in metric spaces [1, Chapter 1]. But not all range spaces of random variables are in a natural way metrizable and we are presenting now a rather general version of Slutsky's theorem which hopefully will cover all situations where a result like this may be needed.

Let E be an arbitrary topological space (not necessarily Hausdorff). The Borel-σ-field of E (i.e. the σ-field generated by the topology) is denoted $\mathcal{B}(E)$. A probability measure μ on $\mathcal{B}(E)$ is called *τ-smooth* if and only if $\mu(G) = \sup_\alpha \mu(G)$ for every net of open subsets $\{G_\alpha\}$ of E filtering up to G. This condition can be regarded as a kind of minimal compatibility of the measure with the underlying topology. If E has a countable base then of course every measure is τ-smooth. If E is completely metrizable or locally compact, then τ-smoothness is equivalent with being a Radon measure, cf. [4, p. 16].

Let $M_\tau^1(E)$ denote the space of all τ-smooth probability measures on some topological space E. Then the weak topology on $M_\tau^1(E)$ is by definition the coarsest topology such that for each bounded lower semicontinuous functions $f\colon E \to \mathbf{R}$ the associated function $\mu \mapsto \int f\, d\mu$ on $M_\tau^1(E)$ is also lower semicontinuous. The famous Portmanteau theorem (cf. [4, Theorem 8.1]) states a lot of equivalent conditions for the weak convergence of nets of probability measures from which we only need, that $\mu_\alpha \to \mu$ weakly if and only if $\liminf \mu_\alpha(G) \geq \mu(G)$ for each open set $G \subseteq E$.

Let E and F be two topological spaces, then always $\mathcal{B}(E) \otimes \mathcal{B}(F) \subseteq \mathcal{B}(E \times F)$. The usual product of two measures μ on $\mathcal{B}(E)$ and ν on $\mathcal{B}(F)$ is only defined on

Received by the editors September 5, 1981.
1980 *Mathematics Subject Classification*. Primary 60B10; Secondary 60B05.

$\mathcal{B}(E) \otimes \mathcal{B}(F)$, but if μ and ν are both τ-smooth, then there is a unique τ-smooth extension of $\mu \otimes \nu$ to the larger σ-field $\mathcal{B}(E \times F)$, denoted $\mu \hat{\otimes} \nu$; cf. [**2**, Theorem 1].

Now we are able to state our result:

THEOREM. *Let E and F be two (not necessarily Hausdorff) spaces. Let $\{\rho_\alpha\}$ be a net of τ-smooth probability measures on $E \times F$ with marginals $\{\mu_\alpha\}$ on E and $\{\nu_\alpha\}$ on F. Assume that $\{\mu_\alpha\}$ converges to some τ-smooth measure μ and that $\{\nu_\alpha\}$ converges to a one-point measure ϵ_y, where $y \in F$. Then $\{\rho_\alpha\}$ converges to $\mu \hat{\otimes} \epsilon_y$.*

PROOF. Let $G \subseteq E \times F$ be open, then there are open sets $U_\lambda \subseteq E$, $V_\lambda \subseteq F$, $\lambda \in \Lambda$ such that $G = \bigcup_{\lambda \in \Lambda}(U_\lambda \times V_\lambda)$. The measure $\rho := \mu \hat{\otimes} \epsilon_y$ being τ-smooth, we can find, given $\epsilon > 0$, finitely many $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that

$$\rho\left(\bigcup_{j=1}^{n}(U_{\lambda_j} \times V_{\lambda_j})\right) > \rho(G) - \epsilon.$$

Put $G_0 := \bigcup_{j=1}^{n}(U_{\lambda_j} \times V_{\lambda_j})$ and $I := \{j \leq n : y \in V_{\lambda_j}\}$. If $I = \emptyset$, then $\rho(G_0) = 0$ and certainly $\liminf \rho_\alpha(G_0) \geq \rho(G_0)$. Suppose now that $I \neq \emptyset$. Let $A := \bigcup_{j \in I} U_{\lambda_j}$ and $B := \bigcap_{j \in I} V_{\lambda_j}$; then $G_1 := A \times B \subseteq G_0$ and $\rho(G_1) = \rho(G_0)$. Using the fact that

$$\limsup \nu_\alpha(B^c) \leq \epsilon_y(B^c) = 0$$

we get

$$\begin{aligned}\liminf \rho_\alpha(G_0) &\geq \liminf \rho_\alpha(G_1)\\ &\geq \liminf(\mu_\alpha(A) - \nu_\alpha(B^c))\\ &\geq \mu(A) = \rho(G_0) > \rho(G) - \epsilon.\end{aligned}$$

Hence

$$\liminf \rho_\alpha(G) \geq \rho(G)$$

and the above-mentioned lim inf-condition shows that $\rho = \lim \rho_\alpha$. $\square$

We now show how easily the classical theorem of Slutsky mentioned in the introduction (and a lot of similar results) are obtained from our theorem.

Suppose that X_n, Y_n, $n = 1, 2, \ldots$, are real random variables such that P^{X_n} (the distribution of X_n) converges to μ, Y_n converging in probability (and therefore in distribution, too) to zero. Let $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$ be defined by $\varphi(x, y) := x + y$. Then

$$P^{X_n + Y_n} = P^{\varphi(X_n, Y_n)} = (P^{X_n, Y_n})^\varphi$$

and from our theorem

$$P^{(X_n, Y_n)} \to \mu \otimes \epsilon_0.$$

The continuity of φ now implies

$$P^{X_n + Y_n} \to (\mu \otimes \epsilon_0)^\varphi = \mu * \epsilon_0 = \mu.$$

If instead Y_n converges to one, we have to replace φ by the continuous function $\psi(x, y) := xy$ to get

$$P^{X_n Y_n} \to \mu.$$

The extension of Theorem 1 to the product of more than two, but finitely many spaces is more or less obvious. The following lemma shows that the result also extends to countable products.

LEMMA. *Let $E_1, E_2, \ldots$ be a countable sequence of topological spaces whose product is denoted E. Let $\{\rho_\alpha\}$ be a net of τ-smooth probability measures on E such that $\{\rho_\alpha^i\}$, the net of marginals in the ith coordinate, converges to a one-point measure ϵ_{x_i} for each $i \geq 1$. Then $\{\rho_\alpha\}$ converges to ϵ_x, where $x := (x_1, x_2, \ldots)$.*

PROOF. Let π_n: $E \to \prod_{i=1}^n E_i$ be the natural projection. By induction we conclude from Theorem 1 that $\{\rho_\alpha^{\pi_n}\}$ converges to $\epsilon_{(x_1,\ldots,x_n)}$ for all $n \in \mathbf{N}$. If $G \subseteq E$ is open, then G has the form

$$G = \bigcup_{n=1}^{\infty} \pi_n^{-1}(G_n)$$

where $G_n \subseteq \prod_{i=1}^n E_i$ is open for all n.

We have to show $\liminf \rho_\alpha(G) \geq \epsilon_x(G)$ and this is obvious if $x \notin G$. But if $x \in G$ then $x \in \pi_n^{-1}(G_n)$ for some n and hence

$$\begin{aligned} \lim \rho_\alpha(G) &\geq \liminf \rho_\alpha(\pi_n^{-1}(G_n)) \\ &= \liminf \rho_\alpha^{\pi_n}(G_n) \\ &\geq \epsilon_{(x_1,\ldots,x_n)}(G_n) = 1. \quad \square \end{aligned}$$

REFERENCES

1. P. Billingsley, *Convergence of probability measures*, Wiley, New York, 1968.
2. P. Ressel, *Some continuity and measurability results on spaces of measures*, Math. Scand. **40** (1977), 69–78.
3. E. E. Slutsky, *Über stochastische Asymptoten und Grenzwerte*, Metron **5** (1925), 1–90.
4. F. Topsøe, *Topology and measure*, Lecture Notes in Math., vol. 133, Springer-Verlag, Berlin and New York, 1970.

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