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WHEN DO THE SYMMETRIC TENSORS OF A COMMUTATIVE ALGEBRA FORM A FROBENIUS ALGEBRA?

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ABSTRACT. For a commutative k-algebra B, consider the subalgebra $(B^{\otimes n})^{S_n}$ of the nth tensor power of B, formed by the tensors invariant under arbitrary permutations of the indices. Necessary and sufficient conditions are found for $(B^{\otimes n})^{S_n}$ to be Frobenius. When $\dim_k B \neq 2$, these say that B is Frobenius and $n!$ is invertible in k, unless B is separable. Some additional cases occur for two-dimensional algebras in positive characteristic, depending on the divisibility of $n+1$.

1. Introduction. Let k be a field, and let B be a commutative k-algebra. In [**1**] G. Azumaya considers the following question, raised by N. Jacobson: If B is Frobenius and G is the group of order two generated by τ, acting on $A = B \otimes B$ by $\tau(b_1 \otimes b_2) = b_2 \otimes b_1$, is the fixed subalgebra A^G also Frobenius? (Recall that by definition, B is Frobenius if it is finite-dimensional over k, and $B \simeq \operatorname{Hom}_k(B, k)$ as B-modules.) The main result of [**1**] shows that when B is generated over k by a single element, then A^G is Frobenius if and only if the characteristic of k is different from 2, or B is a direct product of separable field extensions. As an immediate corollary to our theorem below, we obtain a complete answer to Jacobson's question, by showing that the restriction on the number of generators is superfluous.

More generally, in this note we are concerned with the subalgebra of the n-fold k-tensor product $B^{\otimes n}$, consisting of symmetric tensors, i.e. the fixed subalgebra of $B^{\otimes n}$ under the action of the symmetric group S_n:

$$\tau(b_1 \otimes \cdots \otimes b_n) = b_{\tau^{-1}(1)} \otimes \cdots \otimes b_{\tau^{-1}(n)}$$

for $\tau \in S_n$. Our result gives a complete description of those B and n, for which $(B^{\otimes n})^{S_n}$ is Frobenius. In what follows, all algebras, unless expressly stated, are over k, and have a unit which is preserved by algebra homomorphisms; all unadorned tensor products are taken over k.

THEOREM. *For a commutative algebra B, over a field k of characteristic p, the symmetric tensors $(B^{\otimes n})^{S_n}$ form a Frobenius algebra if and only if*
(a) *B is Frobenius, and*
(b) *one of the following (disjoint) conditions holds:*
(i) $p = 0$;
(ii) $p > n$;

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(iii) $0 < p \leq n$, $B \simeq k[X]/(X^2)$, *and* $n = p^j b - 1$ *for some integers* $j \geq 1$ *and* $1 \leq b \leq p-1$;
(iv) $p = 2$, B *is a quadratic purely inseparable field extension of* k, *and* $n = 2^j - 1$ *for some integer* $j \geq 2$;
(v) $0 < p \leq n$, *and* B *is the direct product of (a finite number of finite) separable field extensions.*

REMARK. That condition (a) is necessary for $(B^{\otimes n})^{S_n}$ to be Frobenius is almost immediate, and that (a) together with one of (i)–(v) is sufficient for this to hold, is rather easily seen. Hence, the bulk of our work below goes into showing that when B and $(B^{\otimes n})^{S_n}$ are both Frobenius, one of the conditions from (i)–(v) must be satisfied. However, for reasons of exposition, we have chosen a somewhat different arrangement of the material below.

REMARK. In commutative algebra, one usually speaks, meaning the same thing, of Gorenstein algebras, finite-dimensional over k, rather than of Frobenius ones. Hence a natural extension of the question raised in the title would be: When do the symmetric tensors of a finitely-generated commutative k-algebra form a Gorenstein ring?

That a rather different answer should be expected is indicated by the following immediate corollary of Watanabe's main result in [**2**.II]: Suppose $p = 0$, or $p > n$, and let $B = k[X_1, \ldots, X_m]$ be the polynomial ring in m indeterminates; then $(B^{\otimes n})^{S_n}$ is Gorenstein if and only if either $m = 1$, or m is even.

2. Preliminaries.

(1) LEMMA. *The k-vector spaces B and $(B^{\otimes n})^{S_n}$ are finite-dimensional or not alike. Moreover, if $e_1, \ldots, e_m$ is a k-basis of B, then a k-basis of $(B^{\otimes n})^{S_n}$ is given by the elements*

$$\mathbf{o}_n[j_1, j_2, \ldots, j_m] = \sum a_{\tau^{-1}(1)} \otimes \cdots \otimes a_{\tau^{-1}(n)}$$

where the summation is taken over a set of representatives of the left cosets of $S_n/S_{j_1} \times \cdots \times S_{j_m}$, with $j_1 + \cdots + j_m = n$, and $a_j = e_h$ for $j_1 + \cdots + j_{h-1} < j \leq j_1 + \cdots + j_h$, for $1 \leq h \leq m$. In particular, if $\dim_k B = m$, then $\dim_k (B^{\otimes n})^{S_n} = \binom{m+n-1}{n}$.

PROOF. Left to the reader.

(2) LEMMA. *Let $B = B_1 \times \cdots \times B_t$ be a decomposition as a direct product of k-algebras. Then as k-algebras*

$$(B^{\otimes n})^{S_n} \simeq \prod_{\substack{r_1 + \cdots + r_t = n \\ r_i \geq 0}} (B_1^{\otimes r_1})^{S_{r_1}} \otimes \cdots \otimes (B_t^{\otimes r_t})^{S_{r_t}}.$$

PROOF. Consider the k-linear map f from the right side to the left one, obtained by composing the inclusion of each factor into the corresponding $(B_1^{\otimes r_1}) \otimes \cdots \otimes (B_t^{\otimes r_t}) = B_{r_1, \ldots, r_t}$, with the map sending $b_1 \otimes \cdots \otimes b_n \in B_{r_1, \ldots, r_t}$ to $\sum b_{\tau^{-1}(1)} \otimes \cdots \otimes b_{\tau^{-1}(n)}$, the summation being taken over a set of representatives of the left cosets of $S_n/S_{r_1} \times \cdots \times S_{r_t}$. Clearly, f is injective. To see that it is an isomorphism, suppose first that $\dim B_i = m_i < \infty$ for $1 \leq i \leq t$. By (1) the dimension of the left side is $\binom{m+n-1}{n}$, while that of the right one is $\sum_{r_1+\cdots+r_t=n}\binom{m_1+r_1-1}{r_1}\cdots\binom{m_t+r_t-1}{r_t}$. These integers coincide—compare the coefficient of X^n in the identity of formal power series

$$(1-X)^{-m}=\prod_{i=1}^{t}(1-X)^{-m_i},$$

hence f is bijective. To treat the general case, note that if $B'=B'_1\times\cdots\times B'_t$ ($B'_i\subset B_i$) is a finite-dimensional subspace of B, then f maps $\prod_{r_1+\cdots+r_t=n}(B_1'^{\otimes r_1})^{S_{r_1}}\otimes\cdots\otimes(B_t'^{\otimes r_t})^{S_{r_t}}$ into $(B'^{\otimes n})^{S_n}$, and this is an isomorphism by the case already settled. Since f commutes with direct limits, it is an isomorphism for every B.

It remains to observe that f is a k-algebra map.

(3) COROLLARY. *$(B^{\otimes n})^{S_n}$ is Frobenius if and only if $(B_i^{\otimes r_i})^{S_{r_i}}$ is Frobenius for $1\le i\le t$ and $r_1+\cdots+r_t=n$.*

PROOF. It is clear that a direct product of algebras is Frobenius if and only if each factor is such, and the corresponding property holds, in the commutative case, also for the tensor product: cf. [**3**, Theorem 1′].

3. The local case. Recall that a finite-dimensional commutative local k-algebra $(B,\mathbf{m})$ is Frobenius if and only if $\dim_{B/\mathbf{m}}(0:\mathbf{m})=1$.

(4) LEMMA. *Let $0<p\le n$, and $B=k[X]/(X^2)$. Then $(B^{\otimes n})^{S_n}$ is Frobenius if and only if $n=p^jb-1$, for some integers $j\ge 1$, $1\le b\le p-1$.*

PROOF. Identifying $B^{\otimes n}$ with the graded k-algebra

$$k[X_1,\ldots,X_n]/(X_1^2,\ldots,X_n^2),$$

(1) shows that the elementary symmetric functions σ_q ($1\le q\le n$), together with 1, provide a homogeneous basis for the symmetric tensors. The product rule being $\sigma_r\sigma_q=\binom{r+q}{q}\sigma_{r+q}$, where $\sigma_i=0$ for $i>n$, one sees, using the formula

$$\binom{a}{b}\equiv\prod\binom{a_i}{b_i}\bmod(p)$$

(for $a=\sum a_ip^i$, $b=\sum b_ip^i$, $0\le a_i\le p-1$, $0\le b_i\le p-1$) that $\sigma_i^p=0$ for all i, and that if q is not a power of p, then σ_q is expressible as a product of elements of lower degree. In particular, the elements σ_{p^i}, when $1\le p^i\le n$, generate $(B^{\otimes n})^{S_n}$ as a k-algebra.

Let now j and b be the integers uniquely defined by the inequalities $bp^j-1\le n<(b+1)p^j-1$, ($p\le n$), $1\le b\le p-1$. The element $s=\sigma_1^{p-1}\cdots\sigma_{p^{j-1}}^{p-1}\sigma_{p^j}^{b-1}$ of degree bp^j-1 is in the socle of $(B^{\otimes n})^{S_n}$: $s\sigma_{p^i}$ is divisible by $\sigma_{p^i}^p=0$ for $0\le i\le j-1$, while $\deg(s\sigma_{p^j})=(b+1)p^j-1>n$. Hence, in case $n>bp^j-1$, the symmetric tensors are not a Frobenius algebra, since a nonzero element of degree n gives a linearly independent from s element of the socle. On the other hand, assume $n=bp^j-1$; then $\sigma_{p^j}^b=0$, since its degree is $bp^j>n$. Hence every nonzero product of generators has the form $x=\sigma_1^{a_0}\cdots\sigma_{p^{j-1}}^{a_{j-1}}\sigma_{p^j}^{a_j}$, with $0\le a_i\le p-1$ ($0\le i\le j-1$), $0\le a_j\le b-1$, and these form a k-basis of $(B^{\otimes n})^{S_n}$. Taking $y=\sigma_1^{c_0}\cdots\sigma_{p^j}^{c_j}$ with $c_i=p-1-a_i$ ($0\le i\le j-1$), $c_j=b-1-a_j$, we see that $x\cdot y=s$, hence s generates the socle of $(B^{\otimes n})^{S_n}$, and we are done.

(5) LEMMA. *Let $(B, \mathbf{m})$ be a local commutative Frobenius k-algebra, such that $B/\mathbf{m} \simeq k$, and $\dim_k B \geq 3$. Then $(B^{\otimes n})^{S_n}$ is Frobenius if and only if either $p = 0$, or $p > n$.*

PROOF. Choose a generator s of $(0\colon \mathbf{m})$. Since $B^{\otimes n}$ is a local Frobenius k-algebra, with socle generated by $s \otimes \cdots \otimes s$ (n factors), and this element is fixed by all $\tau \in S_n$, the invariant subring is (local) Frobenius if either $p = 0$, or $p > n$ by an easy lemma of Watanabe [**2**.I, Lemma 4]. Hence, from now on, we suppose $0 < p \leq n$, and fix a basis $e_1, \ldots, e_m$ of B such that $e_1 = 1$, $e_m = s$, $e_{m-1} = t$, where $t \in (0\colon \mathbf{m}^2)\backslash(0\colon \mathbf{m})$ (this is not empty since B is not a field). Note that $te_i = \alpha_i s$ for some $\alpha_i \in k$ $(2 \leq i \leq m)$. Our purpose is to produce a symmetric tensor $s' \neq 0$, not proportional to $s \otimes \cdots \otimes s$, which annihilates every basis element $x = \mathbf{o}_n[j_1, \ldots, j_m]$ with $j_1 < n$. There are several cases to be considered.

Case I. $n + 1 = pq + r$, with $1 \leq r \leq p - 1$.

Set

$$s' = \mathbf{o}_n[0, \ldots, 0, r, n - r]$$

and note that $s'x = 0$ if $j = j_2 + \cdots + j_m > r$. If $j \leq r$, then $s'x$ is a sum of decomposable tensors with $r - j$ entries equal to t, and $n - r + j$ entries equal to s, each summand appearing with a coefficient $\alpha_2^{j_2}\alpha_3^{j_3}\cdots\alpha_m^{j_m}$. Since all these tensors are in the same S_n-orbit, we have by (1)

$$s'x = A_{j_2,\ldots,j_m}\alpha_2^{j_2}\cdots\alpha_m^{j_m}\mathbf{o}_n[0, \ldots, 0, r - j, n - r + j]$$

for some integer $A_{j_2,\ldots,j_m}$, which is equal to the number of times the element

$$w = \mathbf{o}_{r-j}[0, \ldots, 0, r - j, 0] \otimes \mathbf{o}_{n-r+j}[0, \ldots, 0, n - r + j]$$

occurs in $s'x$. Writing

$$s' = \mathbf{o}_{r-j}[0, \ldots, 0, r - j, 0] \otimes \mathbf{o}_{n-r+j}[0, \ldots, 0, j, n - r] + z,$$
$$x = \mathbf{o}_{r-j}[r - j, 0, \ldots, 0] \otimes \mathbf{o}_{n-r+j}[n - r, j_2, \ldots, j_m] + y,$$

note that w appears only from products of elements not in z with elements not in y, and this happens exactly once for each summand in $x - y$. Hence

$$\begin{aligned} A_{j_2,\ldots,j_m} &= (n - r + j)!/((n - r)!j_2!\cdots j_m!) \\ &= (n - r + j)\cdots(n - r + 1)/(j_2!\cdots j_m!) \end{aligned}$$

and this integer is divisible by p, since the numerator contains a factor $n - r + 1 = pq$, while the denominator is prime to p (because $j_i \leq j \leq r < p$ for $2 \leq i \leq m$).

Case IIa. $n + 1 = pq$, and $\dim_k(0\colon \mathbf{m}^2) > 2$.

Choose a second element $u \in (0\colon \mathbf{m}^2)$ such that s, t, u are linearly independent, and choose a basis of B for which $e_{m-2} = u$, in addition to the previous conditions. Let $ue_i = \beta_i s$ $(\beta_i \in k, 2 \leq i \leq m)$, and set

$$s' = \mathbf{o}_n[0, \ldots, 0, 1, p - 1, n - p].$$

Then $s'x = 0$ when $j = j_2 + \cdots + j_m > p$. Otherwise, this product is expressed as a linear combination of decomposable tensors having $n - p + j$ entries equal to s, at most one entry equal to u, and the remaining ones equal to t; also, note that an entry equal to u will appear only in case $j \leq p - 1$. From (1) we deduce that

$$\begin{aligned} s'x = {} & B_{j_2,\ldots,j_m}\alpha_2^{j_2}\cdots\alpha_m^{j_m}\mathbf{o}_n[0, \ldots, 0, 1, p - j - 1, n - p + j] \\ & + \sum_{\substack{h=2 \\ j_h \geq 1}}^{m} C^h_{j_2,\ldots,j_m}\beta_h\alpha_2^{j_2}\cdots\alpha_h^{j_h - 1}\cdots\alpha_m^{j_m}\mathbf{o}_n[0, \ldots, 0, p - j, n - p + j] \end{aligned}$$

where $B_{j_2,\dots,j_m}$ and $C^h_{j_2,\dots,j_m}$ are integers, the first of which is zero when $j = p$. Moreover, $B_{j_2,\dots,j_m}$ is the number of times the summand

$$u \otimes \mathbf{o}_{p-j-1}[0,\dots,0,p-j-1,0] \otimes \mathbf{o}_{n-p+j}[0,\dots,0,n-p+j]$$

appears in $s'x$, and similarly $C^h_{j_2,\dots,j_m}$ is the number of times the summand

$$\mathbf{o}_{p-j}[0,\dots,0,p-j,0] \otimes \mathbf{o}_{n-p+j}[0,\dots,0,n-p+j]$$

comes up with a coefficient $\beta_h \alpha_2^{j_2} \cdots \alpha_h^{j_h-1} \cdots \alpha_m^{j_m}$. Writing

$$s' = u \otimes \mathbf{o}_{p-j-1}[0,\dots,0,p-j-1,0] \otimes \mathbf{o}_{n-p+j}[0,\dots,0,j,n-p] + z,$$
$$x = 1 \otimes \mathbf{o}_{p-j-1}[p-j-1,0,\dots,0] \otimes \mathbf{o}_{n-p+j}[n-p,j_2,\dots,j_m] + y,$$

we conclude as in Case I that

$$B_{j_2,\dots,j_m} = (n-p+j)\cdots(n-p+1)/(j_2!\cdots j_m!),$$

which is divisible by p. On the other hand, from

$$s' = \mathbf{o}_{p-j}[0,\dots,0,p-j,0] \otimes \mathbf{o}_{n-p+j}[0,\dots,0,1,j-1,n-p] + z',$$
$$x = \mathbf{o}_{p-j}[p-j,0,\dots,0] \otimes \mathbf{o}_{n-p+j}[n-p,j_2,\dots,j_m] + y',$$

we see that

$$C^h_{j_2,\dots,j_m} = (n-p+j)\cdots(n-p+1)/(j_2!\cdots(j_h-1)!\cdots j_m!),$$

and this integer also is divisible by p.

Case IIb. $n+1 = pq$, and $\dim_k(0\colon \mathbf{m}^2) = 2$.

Since $(0\colon \mathbf{m}^2)/(0\colon \mathbf{m}) \simeq (\mathbf{m}/\mathbf{m}^2)^*$ from the Frobenius condition, one necessarily has $B \simeq k[X]/(X^m)$, $m \geq 3$. In the basis $e_i = X^{i-1} + (X^m)$, $1 \leq i \leq m$, set

$$s' = \mathbf{o}_n[0,\dots,0,1,p-1,n-p].$$

Clearly, s' annihilates x if either $j_h > 0$ for some $h \geq 4$, or $j_3 > 1$, or $j = j_2 + j_3 > p$. In case $j_h = 0$ for $h \geq 3$, and $j = j_2 \leq p$, one has

$$s'x = D\mathbf{o}_n[0,\dots,0,1,p-j-1,n-p+j] + E\mathbf{o}_n[0,\dots,0,p-j+1,n-p+j-1]$$

where $D = 0$ if $j = p$. Arguing as above, one finds $D = \binom{n-p+j}{j}$ (when $j < p$), and $E = (p-j+1)\binom{n-p+j-1}{j-1}$; both are divisible by p. There is just one more possibility, namely $x = \mathbf{o}_n[n-j,j-1,1,0,\dots,0]$, $j \leq p$. Here

$$s'x = F\mathbf{o}_n[0,\dots,0,p-j,n-p+j],$$

and one obtains for F the value $j \cdot \binom{n-p+j}{j}$, which is a multiple of p.

The proof of the lemma is now complete.

4. Proof of the theorem. It is known that the k-algebra B is Frobenius if and only if the $\overline{k}$-algebra $B \otimes_k \overline{k}$ is Frobenius, $\overline{k}$ being the algebraic closure of k. Moreover, the n-fold tensor product C of $B \otimes_k \overline{k}$ with itself over $\overline{k}$ is isomorphic to $(B^{\otimes n}) \otimes_k \overline{k}$, this isomorphism commutes with the S_n-actions on C and on $(B^{\otimes n}) \otimes_k \overline{k}$, and $C^{S_n} \simeq (B^{\otimes n})^{S_n} \otimes_k \overline{k}$. Hence, we can assume k is algebraically closed.

Suppose $(B^{\otimes n})^{S_n}$ is Frobenius. Then $\dim_k B$ is finite by (1), hence B has a decomposition $B = B_1 \times \cdots \times B_t$ into finite-dimensional local k-algebras with residue field k. Supposing B is not Frobenius, there must be a B_i with two linearly independent elements s_1, s_2 in its socle. But then $s_1 \otimes \cdots \otimes s_1$ and $s_2 \otimes \cdots \otimes s_2$

(n factors each) are linearly independent elements in the socle of $(B_i^{\otimes n})^{S_n}$, hence this algebra is not Frobenius. By (3) this is a contradiction to the hypothesis on $(B^{\otimes n})^{S_n}$. Thus, what remains to be done is, assuming B Frobenius, to show that $(B^{\otimes n})^{S_n}$ is Frobenius if and only if one of the conditions from (i)–(v) (in (b) of the theorem) holds.

By (3), $(B^{\otimes n})^{S_n}$ is Frobenius if and only if this is true for $(B_i^{\otimes r_i})^{S_{r_i}}$ with $1 \leq i \leq t$ and $r_1 + \cdots + r_t = n$. Applying (5), we see that when $\dim_k B_i \geq 3$ for some i, this is fulfilled if and only if either $p = 0$, or $p > n$.

In what follows, we assume $0 < p \leq n$, and $\dim_k B_i \leq 2$ for $1 \leq i \leq t$. When $\dim_k B_i = 2$ for some $i, t \geq 2$ is impossible, since by (3) and (4) one must have $n = p^j b - 1$ and $n - 1 = p^h c - 1$ with $j, h \geq 1$ and $1 \leq b,\, c \leq p - 1$. Hence in this case $B \simeq k[X]/(X^2)$, and $(B^{\otimes n})^{S_n}$ is Frobenius if and only if $n = p^j b - 1$. But all algebras, which after extension of scalars to the algebraic closure, become isomorphic to $k[X]/(X^2)$, are described by conditions (iii) and (iv) of the theorem.

Finally, condition (v) of the theorem (k arbitrary), becomes equivalent, after passing to the algebraic closure, to the equalities $\dim_k B_i = 1$ $(1 \leq i \leq t)$, and in this case

$$(B^{\otimes n})^{S_n} \simeq k^{\binom{n+t-1}{t}}$$

is clearly Frobenius.

REFERENCES

1. G. Azumaya, *Fixed subalgebra of a commutative Frobenius algebra*, Proc. Amer. Math. Soc. **81** (1981), 213–216.
2. K. Watanabe, *Certain invariant subrings are Gorenstein.* I and II, Osaka J. Math. **11** (1974), 1–8, 379–388.
3. K. Watanabe, T. Ishikawa, S. Tachibana and K. Otsuka, *On tensor products of Gorenstein rings*, J. Math. Kyoto Univ. **9** (1969), 413–423.

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