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ON A THEOREM OF FLANDERS

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ABSTRACT. It is shown that if R is a regular strongly-pi-regular ring, then R is unit-regular precisely when $(ab)^d \approx (ba)^d$ for all $a, b \in R$. This generalizes a result by Flanders, which states that the matrices AB and BA over a field $\mathbf{F}$ have the same elementary divisors except possibly those divisible by λ.

1. Introduction. A classic theorem of Flanders [3] states that if A and B are $n \times n$ matrices over a field $\mathbf{F}$, then AB and BA have the same elementary divisors, except possibly for those that are powers of λ.

The purpose of this note is to point out that the real reason why this result is true, is because the matrix ring $\mathbf{F}_{n \times n}$ is both strongly-pi-regular as well as unit-regular. We shall use the concept of pseudo-similarity, introduced in [5] to provide the necessary link between strong-pi-regularity and unit-regularity.

We recall that a ring R is called (unit) regular, if for every $a \in R$, there exists a (unit) solution $x \in R$, to the equation $axa = a$. Such solutions will be denoted at a^-. A ring R is strongly-pi-regular, sπr for short, if for every $a \in R$, there is a solution to the equations

$$a^k xa = a^k, \quad xax = x, \quad ax = xa,$$

for some $k \geq 0$. The solution is unique and is called the Drazin inverse a^d of a [2]. In the special cases where $k = 0$ or $k = 1$, a^d is called the group inverse of a, and is denoted by $a^{\#}$. It is well known that the ring $\mathbf{F}_{n \times n}$ of $n \times n$ matrices over a field $\mathbf{F}$ is both sπr and unit-regular. Two ring elements are called pseudo-similar if

$$x^- ax = b, \quad xbx^- = a, \quad xx^- x = x$$

for some $x, x^- \in R$. It was shown in [6], that for a unit-regular ring, similarity ($\approx$) and pseudo-similarity ($\overline{\approx}$), coincide. Two idempotents e and f in R, are said to be equivalent, $e \sim f$, if eR and fR are isomorphic ($\cong$) as R-modules. This may be rewritten as $e \sim f$ if $e = p^+ p$, $f = pp^+$, for some $p, p^+ \in R$, that satisfy $pp^+ p = p$ and $p^+ pp^+ = p^+$. It is easily seen that $e \sim f \Leftrightarrow e \overline{\approx} f$ [4] and that $\sim$ actually coincides with the classical $\mathcal{D}$-relation on semigroups.

2. Main results. Our generalization of the theorem of Flanders is based on the following two main results.

THEOREM 1. *Let R be a strongly-pi-regular ring with unity and let $x, y \in R$. Then the following are equivalent:*
(i) $x^d \approx y^d$,
(ii) $x^2 x^d \approx y^2 y^d$.
In which case

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(iii) $xx^d \approx yy^d$ *and* $(x^d + 1 - xx^d) \approx (y^d + 1 - yy^d)$.

If in addition $R = \mathbf{F}_{n\times n}$, *then each of the above is equivalent to*

(iv) x *and* y *have the same elementary divisors, except possibly those that are powers of* λ.

PROOF. (i)⇔(ii) This is based on a result of Drazin [**2**], which states that if $pq = qp$, and $p^\#$ exists, then $p^\# q = qp^\#$. Now if $x^d q = qy^d$, with q invertible, then because $(x^d)^\# = x^2x^d$, it follows that $x^2x^dq = qy^2y^d$. The converse follows also since $(p^\#)^\# = p$.

(i)⇒(iii) Let $x^dq = qy^d$, and hence by part (ii) $x^2x^dq = qy^2y^d$. Now $xx^d = (x^2x^d)x^d$ and so $xx^dq = qyy^d$. The remaining result is now also clear.

(iii)⇒(iv) Let $R = \mathbf{F}_{n\times n}$, and suppose that

$$x = P\begin{bmatrix} U_1 & 0 \\ 0 & \eta_1 \end{bmatrix}P^{-1}, \qquad y = Q\begin{bmatrix} U_2 & 0 \\ 0 & \eta_2 \end{bmatrix}Q^{-1}, \tag{2.1}$$

are the core-nilpotent decompositions of x and y respectively, with U_i invertible and η_i nilpotent, $i = 1, 2$. Then

$$x^d = P\begin{bmatrix} U_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1},$$

and $xx^d \approx yy^d$ shows that U_1 and U_2 have the same size. Next,

$$\begin{bmatrix} U_1^{-1} & 0 \\ 0 & I \end{bmatrix} \approx x^d + 1 - xx^d \approx y^d + 1 - yy^d \approx \begin{bmatrix} U_2^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

Hence, on using elementary divisors, it follows that $U_1^{-1} \approx U_2^{-1}$ and so $U_1 \approx U_2$. This is equivalent to (iv).

(iv)⇒(i) This is clear from the fact that $x^d \approx y^d \Leftrightarrow U_1^{-1} \approx U_2^{-1} \Leftrightarrow U_1 \approx U_2$.

Our second result deals with the relation between ab and ba.

THEOREM 2. *Let* R *be a strongly-pi-regular ring with unity, and let* $a, b \in R$. *Then the following are equivalent:*

(i) R *is unit-regular, and*

(ii) R *is regular and* $(ab)^d \approx (ba)^d$ *for all* $a, b \in R$.

PROOF. (i)⇒(ii) Cline's formula [**1**], states that $(xy)^d = x(yx)^{d^2}y$. If we apply this to ab and ba we may write

$$(ab)^d = a \cdot (ba)^d \cdot (ba)^d b$$

and

$$(ba)^d = (ba)^d b \cdot (ab)^d \cdot a.$$

Now since $(ba)^d b \cdot a \cdot (ba)^d b = (ba)^d b$, it follows that $(ab)^d \approx (ba)^d$. Moreover, $ab(ab)^d \approx ba(ba)^d$ as well. Next, since R is assumed to be unit-regular, we may conclude [**6**, p. 453] that semisimilarity implies similarity. Consequently $(ab)^d \approx (ba)^d$ and $ab(ab)^d \approx ba(ba)^d$.

(ii)⇒(i) Based on a result of Vidav [**7**], it suffices to show that two equivalent idempotents are similar. Suppose therefore that e and f are equivalent idempotents in R, and that $e = p^+pf = pp^+$, with $pp^+p = p$ and $p^+pp^+ = p^+$. Now since $e^d = e$ and $f^d = f$, we have $e = e^d = (p^+p)^d \approx (pp^+)^d = f^d = f$. Because R is regular, the proof is now complete.

COROLLARY 1. *If* $A, B \in \mathbf{F}_{n\times n}$, *then*
(i) $(AB)^d \approx (BA)^d$,
(ii) $(AB)^2(AB)^d \approx (BA)^2(BA)^d$, *and*
(iii) *the elementary divisors of AB and BA coincide except possibly for those that are powers of* λ.

PROOF. (i) Since $\mathbf{F}_{n\times n}$ is unit-regular and sπr, this follows from Theorem 2. The remaining two results follow at once from Theorem 1, thereby completing the proof of Flander's Theorem.

COROLLARY 2. *If* $A, B \in \mathbf{F}_{n\times n}$, *then* $AB \approx BA$ *if and only if* $\operatorname{rank}(AB)^k = \operatorname{rank}(BA)^k$, $k = 1, 2, \ldots$.

PROOF. The necessity is clear. For the sufficiency, suppose that $AB = x$ and $BA = y$ are given as in (2.1). It then suffices to show that $U_1 \approx U_2$ and $\eta_1 \approx \eta_2$. From Corollary 1, we recall that $U_1 \approx U_2$ always holds. Lastly, $\operatorname{rank}(AB)^k = \operatorname{rank}(BA)^k$ implies that $\operatorname{rank}(\eta_1^k) = \operatorname{rank}(\eta_2)^k$, for all $k = 1, 2, \ldots$, which is well known to suffice for $\eta_1 \approx \eta_2$.

REMARKS. 1. The results of Corollary 1 can easily be modified to include the case where A and B are rectangular, by means of adding zeros suitably.

2. The equivalence of the pseudo-similarity and similarity of $(ab)^d$ and $(ba)^d$, hinges on the existence of a unit solution to the equation $(ba)^d b \cdot x \cdot (ba)^d b = (ba)^d b$. Even though several obvious solutions exist, such as $x = a$ or $x = a + 1 - (ba)(ba)^d$, it is not known whether there exists a unit solution u that can be expressed in terms of a, b and $(\cdot)^d$ exclusively.

3. It is not known whether the converse of Theorem 1 holds in general. That is whether $xx^d \approx yy^d$ and $x^d + 1 - xx^d \approx y^d + 1 - yy^d$ ensure that $x^d \approx y^d$.

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