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# THE LATTICE OF LEFT IDEALS IN A CENTRALIZER NEAR-RING IS DISTRIBUTIVE

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ABSTRACT. A decomposition theorem for a left ideal in a finite centralizer near-ring is established. This result is used to show that the lattice of left ideals in a finite centralizer near-ring is distributive.

**1. Introduction.** In the development of a density theorem for 2-primitive near-rings with identity, as presented by Betsch in [**1**], a key lemma for the proof of the density theorem is Lemma 2.9 of [**1**] due to Wielandt [**6**].

LEMMA (WIELANDT). *Let $N$ be an arbitrary near-ring and let $B$, $C$, $D$ be $N$-submodules of some $N$-module. Then the $N$-module*

$$\Gamma = \frac{(B+D)\cap(C+D)}{(B\cap C)+D}$$

*is commutative, and for all $n \in N$ the mapping $\Gamma \to \Gamma$ defined by $\gamma \to n(\gamma)$ is an endomorphism of $(\Gamma, +)$.*

An immediate consequence of Wielandt's lemma is the following found in [**1**].

COROLLARY. *Let $N$ be a near-ring with identity such that no nonzero homomorphic image of $N$ is a ring, then the lattice of left ideals of $N$ is distributive, that is $(B+D)\cap(C+D) = (B\cap C)+D$ for any left ideals $B$, $C$, $D$ of $N$.*

Thus in near-rings $N$ that satisfy the hypothesis of the corollary, the lack of elementwise left distributivity in $N$ is compensated for by a gain in the distributivity of left ideals.

It is natural to ask which near-rings have the property that their lattice of left ideals is distributive. It is the goal of this paper to show that if $N$ is a finite centralizer near-ring then the lattice of left ideals of $N$ is distributive. Since such a near-ring can have a nonzero ring as a homomorphic image (see [**4**]), this result does not follow from the corollary to Wielandt's lemma.

We begin by recalling the definition of a centralizer near-ring. Let $(G, +)$ be a group with identity 0 and $A$ a group of automorphisms of $G$. The centralizer near-ring determined by $G$ and $A$ is the set

$$C(A;G) = \{f\colon G \to G \mid f\alpha = \alpha f \text{ for all } \alpha \in A, f(0) = 0\},$$

forming a near-ring under function addition and function composition. Centralizer near-rings arise naturally in the classification of 2-primitive near-rings [**5**, Chapter 4] and play a role in near-ring theory analogous to that of matrix rings in ring

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theory. In this paper we deal only with finite centralizer near-rings, that is $(G,+)$ is a finite group.

We now establish some concepts and notations used throughout this paper in relation to the centralizer near-ring $N = C(A;G)$. For $v \in G$ we denote by stab$(v)$ the stabilizer subgroup $\{\alpha \in A | \alpha v = v\}$ of $A$ and by $\theta(v)$ the $A$-orbit of $G$ containing $v$. Two orbits $\theta(w)$, $\theta(v)$ are *synonymous*, written $\theta(w) \sim \theta(v)$, if there exist $w' \in \theta(w)$, $v' \in \theta(v)$ with stab$(w')$ = stab$(v')$. The set of all orbits of $G$ is partially ordered as follows: $\theta(w) < \theta(v)$ if and only if there exist $w' \in \theta(w)$, $v' \in \theta(v)$ such that stab$(w') \supset$ stab$(v')$ (proper containment). We will use the notation $\theta(w) \lesssim \theta(v)$ to mean $\theta(w) < \theta(v)$ or $\theta(w) \sim \theta(v)$. Similarly the elements of $G$ are partially ordered as follows: $w < v$ if and only if stab$(w) \supset$ stab$(v)$ (proper containment), and $w \sim v$ if and only if stab$(w)$ = stab$(v)$. Finally $w \lesssim v$ means stab$(w) \supseteq$ stab$(v)$. It is easy to see that $w \lesssim v$ if and only if there exists an element $f \in C(A;G)$ such that $f(v) = w$, a result due to G. Betsch (at the 1976 Oberwolfach Conference on near-rings).

Throughout this article $\theta(v_1), \theta(v_2), \ldots, \theta(v_n)$, $\{0\}$ are assumed to be the $A$-orbits of the finite group $G$. The orbit representatives $v_1, \ldots, v_n$ are assumed to have the property that if $\theta(v_i) \lesssim \theta(v_j)$ then $v_i \lesssim v_j$. A function $f \in C(A;G)$ is completely determined once its action on each $v_i$ is known. In analogy with matrix units in complete matrix rings we define the following special functions on $G$ which belong to $C(A;G)$. For $i = 1, \ldots, n$ let $e_i \colon G \to G$ be the identity on $\theta(v_i)$ and zero off $\theta(v_i)$. Each $e_i$ is idempotent and $1 = e_1 + \cdots + e_n$. For orbits $\theta(v_i)$, $\theta(v_j)$ with $\theta(v_i) \lesssim \theta(v_j)$ define $e_{ij} \colon G \to G$ by $e_{ij}(v_j) = v_i$ and $e_{ij}$ is zero off $\theta(v_j)$.

**2. Decomposition of left ideals.** In this section we derive a decomposition theorem for left ideals $L$ in $C(A;G)$ which will be used in the final section to prove that the left ideals of $C(A;G)$ form a distributive lattice.

LEMMA 1. *Suppose $L$ is a left ideal of $C(A;G)$ and let $\theta(v_k)$, $\theta(v_j)$ be orbits of $G$ under $A$ with $v_k < v_j$. If there exists an $f \in L$ such that $f(v_j) \in \theta(v_k)$ and $f(v_j) + v_j \in \theta(v_k)$, then $e_j \in L$.*

PROOF. Since $e_k f \in L$ we may assume the range of $f$ is $\theta(v_k) \cup \{0\}$. Let $g = e_k(f+e_j) - e_k e_j = e_k(f+e_j)$, an element in $L$. We have $g(v_j) = e_k(f(v_j)+v_j) = f(v_j) + v_j$, and $g(x) = f(x)$ for $x \notin \theta(v_j)$. So $-f+g \in L$ and $(-f+g)(v_j) = -f(v_j) + f(v_j) + v_j = v_j$, $(-f+g)(x) = 0$, $x \notin \theta(v_j)$. Hence $-f+g = e_j \in L$.

LEMMA 2. *Suppose $L$ is a left ideal of $C(A;G)$ and let $\theta(v_i)$ be an orbit of $G$ under $A$. If $f \in L$ is such that $f(v_i) \sim v_i$ then $e_i \in L$.*

PROOF. We may assume $f(v_i) = v_i$. For if $f(v_i) \in \theta(v_j)$ then $\theta(v_j) \sim \theta(v_i)$ and $e_{ij} \in C(A;G)$. Also $e_{ij} f \in L$ with $e_{ij} f(v_i) \in \theta(v_i)$. Moreover some power of $e_{ij} f$ is the identity on $\theta(v_i)$.

As in the proof of Lemma 1 we may also assume that the range of $f$ is $\theta(v_i) \cup \{0\}$. Hence if $f(v_k) \neq 0$ for some $k \neq i$, then $f(v_k) = \beta_k v_i$, $\beta_k \in A$.

Finally we may assume $f$ is nonzero off $\theta(v_i)$, for otherwise $f = e_i$ and we are done. Among all such $f \in L$, select $f$ so that the number of such orbits $\theta(v_k)$ for which $f(v_k) \neq 0$ is minimal. Suppose $f(v_k) = \beta_k v_i$, $k \neq i$.

*Case* 1. Assume there exists a $w \in G$ such that $w \neq 0$, $w \lesssim v_i$, $w \notin \theta(v_i)$ and $v_i + w \notin \theta(v_i)$. Let $g$ be the element in $C(A;G)$ with $g(v_i) = 0$, $g(v_k) = \beta_k w$ and

$g(x) = 0$ if $x \notin \theta(v_i) \cup \theta(v_k)$. Then $e_i(f+g) - e_i g \in L$ and $e_i(f+g) - e_i g = e_i$ due to the minimality of $f$. Hence $e_i \in L$ as desired.

*Case* 2. Assume $v_i + w \in \theta(v_i)$ for every $w$ such that $w \lesssim v_i$, $w \notin \theta(v_i)$. In this case we claim $\theta(v_i)$ is synonymous only to itself. For suppose $\theta(v_i) \sim \theta(v_k)$, yet $\theta(v_i) \neq \theta(v_k)$ where $v_i \sim v_k$. Let $\alpha_1 v_i = v_i, \alpha_2 v_i, \ldots, \alpha_t v_i$ be the distinct elements of $\theta(v_i)$ having the same stabilizer as $v_i$, that is $\alpha_j v_i \sim v_i$, $j = 1, 2, \ldots, t$. Then since $\theta(v_i) \sim \theta(v_k)$, $\alpha_1 v_k = v_k, \alpha_2 v_k, \ldots, \alpha_t v_k$ are the distinct elements of $\theta(v_k)$ which are synonymous to $v_i$. By assumption $v_i + \alpha_j v_k \in \theta(v_i)$ for $j = 1, 2, \ldots, t$. Moreover these elements are all distinct and $v_i + \alpha_j v_k \sim v_i$ for all $j$. But none is equal to $v_i$, so $\theta(v_i)$ contains $t+1$ elements $v_i$, $v_i + v_k, \ldots, v_i + \alpha_t v_k$ synonymous with $v_i$. This contradicts $\theta(v_i)$ having $t$ such elements. Hence $\theta(v_i)$ is a unique orbit type as claimed.

We now have that if $f(v_k) = \beta_k v_i$ for some $k \neq i$ then $v_i < v_k$. If $\beta_k v_i + v_k \notin \theta(v_i)$ then $e_i(f + e_k) - e_i e_k = e_i$ due to the minimality of $f$. So $e_i \in L$. If $\beta_k v_i + v_k \in \theta(v_i)$, then Lemma 1 applies and $e_k \in L$. This means $f - f e_k = e_i \in L$, due to the minimality of $f$.

THEOREM 1. *Let $L$ be a left ideal of $C(A;G)$. Then for each orbit $\theta(v_i)$ of $G$ under $A$, $Le_i \subseteq L$.*

PROOF. Select $f \in L$. If $f(v_i) = 0$ then $fe_i = 0 \in L$, so we may assume $f(v_i) = w \in \theta(v_k)$. We have $e_k f \in L$ and $e_k f e_i = f e_i$. Thus we may assume the range of $f$ is contained in $\theta(v_k) \cup \{0\}$. If $f$ is zero off $\theta(v_i)$ then $fe_i = f \in L$ and we are done. As in the proof of Lemma 2 we may reselect $f$ so that it agrees with the original function on $\theta(v_i)$ and is nonzero on a minimal number of orbits. Selecting $x \notin \theta(v_i)$ such that $f(x) \neq 0$ means $f(x) = \alpha v_k$ for some $\alpha \in A$. Since $w \in \theta(v_k)$, $x$ may be selected so that $x \gtrsim w$.

*Case* 1. Assume $x > w$. We have $f(x) = \alpha v_k$. If $f(x) + x = \alpha v_k + x \notin \theta(v_k)$, then $e_k(f + e_x) - e_k e_x = f e_i$ due to the minimality of $f$. So in this situation $fe_i \in L$. Assume now that $f(x) + x \in \theta(v_k)$. Let $g = e_k(f + e_x) - e_k e_x$. Then $g(x) = f(x) + x$ and $g = f$ off $\theta(x)$. We have $g \in L$ and $(-f + g)(x) = -f(x) + f(x) + x = x$ and $-g + f$ is zero off $\theta(x)$. Hence $-f + g = e_x \in L$. So $f - fe_x = fe_i \in L$, again using the minimality of $f$.

*Case* 2. Assume $x \sim w$. Then $f(x) = \alpha v_k$ for some $\alpha \in A$. Hence $e_x \in L$ by Lemma 2 and $f - fe_x = fe_i \in L$, again using the minimality of $f$.

COROLLARY. *Let $L$ be a left ideal of $C(A;G)$. Then $L = Le_1 \oplus \cdots \oplus Le_n$.*

PROOF. From the theorem, $Le_1 + \cdots + Le_n \subseteq L$. Also if $f \in L$ then $f = fe_1 + \cdots + fe_n$. Thus $L = Le_1 \oplus \cdots \oplus Le_n$ since

$$Le_i \cap (Le_1 + \cdots + Le_{i-1} + Le_{i+1} + \cdots + Le_n) = \{0\}.$$

**3. The lattice of left ideals of $C(A;G)$ is distributive.** Let $L$ and $L'$ be left ideals of $C(A;G)$. From the corollary to Theorem 1, $L = \sum Le_i$ and $L' = \sum L'e_i$. We have

(1) $L = L'$ iff $Le_i = L'e_i$ for every $i$,
(2) $L + L' = \sum (L + L')e_i$,
(3) $L \cap L' = \sum (L \cap L')e_i$.

Now let $B = \sum Be_i$, $C = \sum Ce_i$ and $D = \sum De_i$ be left ideals of $C(A; G)$. Using properties (2) and (3) above we have

$$\begin{aligned}(B + D) \cap (C + D) &= \sum (Be_i + De_i) \cap (Be_i + De_i)\\ &= \sum ((B + D) \cap (C + D))e_i,\\ (B \cap C) + D &= \sum (Be_i \cap Ce_i) + De_i\\ &= \sum ((B \cap C) + D)e_i.\end{aligned}$$

Using property (1) we have established the following lemma.

LEMMA 3. *Let $B$, $C$ and $D$ be left ideals of $C(A; G)$. Then $(B+D) \cap (C+D) = (B \cap C) + D$ if and only if $(Be_i + De_i) \cap (Ce_i + De_i) = (Be_i \cap Ce_i) + De_i$ for $i = 1, \ldots, n$.*

We note that $Be_i$, $Ce_i$, $De_i$ are left ideals of $N = C(A; G)$ contained in the left ideal $Ne_i$. Lemma 3 implies that the lattice of left ideals of $N = C(A; G)$ is distributive provided the lattice of left ideals of $N$ contained in $Ne_i$ is distributive for $i = 1, \ldots, n$.

For each $i$ let $T(v_i) = \{w \in G | w \lesssim v_i\}$, a subgroup of $G$. For $y \in G$ let $P(y; v_i) = \{w \in \theta(y) | w \lesssim v_i\}$. The following result whose proof can be found in [**3**] has relevance to our problem.

THEOREM 2. *Let $N = C(A; G)$ with $v_i \in G^*$, $G^* \equiv G - \{0\}$. Then there exists a one-to-one correspondence between left ideals $L$ of $N$ contained in $Ne_i$ and subsets $H$ of $G$ such that*

(i) *$H$ is a normal subgroup of $T(v_i)$,*
(ii) *$H$ is $N$-invariant,*
(iii) *$P(y; v_i)$ is a union of cosets of $H$ for all $y \in T(v_i) - H$,*
(iv) *if $y \in T(v_i) - H$, $\alpha \in A$ such that $\alpha y - y \in H$ then $\alpha z - z \in H$ for all $z \in T(v_i)$ with $\operatorname{stab}(z) \supseteq \operatorname{stab}(y)$.*

The correspondence mentioned in Theorem 2 is given by $L \to H_L$ where $H_L = \{w | w = f(v_i) \text{ for some } f \in L\} \equiv Lv_i$.

LEMMA 4. *Suppose $L_1$ and $L_2$ are left ideals of $N = C(A; G)$ contained in $Ne_i$. Then either $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$.*

PROOF. Suppose $L_1$, $L_2$ are such that $L_1 \not\subseteq L_2$ and $L_2 \not\subseteq L_1$. We have $L_1 \to H = L_1 v_i$ and $L_2 \to K = L_2 v_i$. Since $L_1 \not\subseteq L_2$ then $H \not\subseteq K$ and since $L_2 \not\subseteq L_1$ then $K \not\subseteq H$. Also $L_1 + L_2 \to H + K$. Select $\tilde{h} \in H$, $\tilde{k} \in K$ such that $\tilde{h} + \tilde{k} \notin H$ and $\tilde{h} + \tilde{k} \notin K$. Since $\tilde{h} + \tilde{k} \in H + K$ there exists an $f \in L_1 + L_2$ such that $f(v_i) = \tilde{h} + \tilde{k}$. We have $f(v_i) \in T(v_i) - K$ so by Theorem 2, part (iii), $P(f(v_i); v_i)$ is a union of cosets of $K$. This means $P(f(v_i); v_i) \supseteq f(v_i) + K = \tilde{h} + \tilde{k} + K$ and so $\tilde{h} \in P(f(v_i); v_i)$.

Also $f(v_i) \in T(v_i) - H$ and by Theorem 2, part (iii), $P(f(v_i); v_i)$ is a union of cosets of $H$. But $\tilde{h} \in P(f(v_i); v_i)$, so $P(f(v_i); v_i) \supseteq \tilde{h} + H = H$. This means $0 \in P(f(v_i); v_i)$, a contradiction to the definition of $P(f(v_i); v_i)$.

THEOREM 3. *The lattice of left ideals of $N = C(A; G)$ is distributive.*

PROOF. From Lemma 3 it suffices to prove that the lattice of left ideals of $N$ contained in $Ne_i$ is distributive for each $i$. From Lemma 4 the left ideals of $N$ contained in $Ne_i$ form a chain and hence the lattice is distributive (see [**2**, p. 441]).

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