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ON SIMPLE REDUCIBLE DEPTH-TWO LIE ALGEBRAS WITH CLASSICAL REDUCTIVE NULL COMPONENT

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ABSTRACT. We classify the simple finite-dimensional reducible graded Lie algebras of the form $L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_k$ over an algebraically closed field of characteristic greater than 3, where L_0 is reductive and classical such that no nonzero element of the center of L_0 annihilates L_{-2} and where L_{-1} is the sum of two proper L_0-submodules.

In [2], the present author showed that if the null component of a simple finite-dimensional Lie algebra of the form

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_k \tag{1}$$

contains no nonzero abelian ideal which annihilates L_{-2}, and if $L_{-1} = S + T$, where S and T are proper L_0-submodules of L_{-1}, then S and T are abelian irreducible L_0-submodules, and $L_{-1} = S \oplus T$. In addition, L possesses an irreducible transitive gradation of the form

$$L = M_{-1} \oplus M_0 \oplus \cdots \oplus M_{[k/2]}, \tag{2}$$

such that (by interchanging the names of S and T if necessary) $M_0 = T \oplus L_0 \oplus [L_2, S]$ and $[L_2, S] = \{0\}$ if k is odd.

In this paper, we strengthen the hypotheses on L_0; we assume that L_0 is classical and reductive and that $\mathrm{Ann}_{L_0} L_{-2}$ contains no nonzero element of the center of L_0. We then prove that L is either classical or of Cartan type. Specifically, we prove the following

THEOREM. *Let L be a simple finite-dimensional reducible graded Lie algebra of the form* (1) *over an algebraically closed field of characteristic greater than* 3, *and suppose that L_0 is a classical reductive Lie algebra no nonzero element of whose center annihilates L_{-2}, and that $L_{-1} = S + T$, where S and T are proper L_0-submodules of L_{-1}. Then L is classical or of Cartan type.*

(For definitions of the terms "transitive" and "irreducible" as they are used in this paper, see [1].)

The proof of this theorem will be carried out by means of a series of lemmas. We begin by recalling certain of the lemmas of [2] for which we will have use here. The numbers in parentheses after the first 10 lemma numbers are those by which the lemmas were designated in [2]. Proofs of these lemmas can be found in [2] (or [1]). In the first 4 lemmas, we assume only that L is a simple graded Lie algebra of the form (1).

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LEMMA 1(1). *L_{-2} is an irreducible L_0-module. Furthermore, $[L_{-1}, L_{-1}] \neq \{0\}$, so $[L_{-1}, L_{-1}] = L_{-2}$; also, $L_{-1} = [L_{-2}, L_1]$. Lastly, $[L_{-1}, x] \neq \{0\}$ for all $x \notin L_{-2}$.*

LEMMA 2(2). *L_k is an irreducible L_0-module, and $L_{j-1} = [L_j, L_{-1}]$ for all j, $-1 \leq j \leq k$.*

LEMMA 3(3). *We have that $[L_0, L_k] \neq \{0\}$ and that $[L_{-2}, L_i] \neq \{0\}$ for all i, $0 \leq i \leq k$.*

LEMMA 4(4). *L_{-1} contains no proper nonabelian L_0-submodule.*

LEMMA 5(6). $[S, [S, L_2]] = \{0\}$.

LEMMA 6(9). $\mathrm{Ann}_{L_i} L_{-2} = \{0\}$ *for all i, $1 \leq i \leq k$.*

LEMMA 7(10). *$L_{-1} = S \oplus T$, where S and T are irreducible abelian L_0-submodules of L_{-1}.*

LEMMA 8(13). $L_{2i-1} = [L_{2i}, S] \oplus [L_{2i}, T]$ *for all i, $0 \leq i \leq [k/2]$.*

LEMMA 9(14). $\mathrm{Ann}_{L_{2i-1}} S = [L_{2i}, S]$ *for all i, $0 \leq i \leq [k/2]$.*

LEMMA 10(15). $[[L_{2i}, S], [L_{2j}, S]] = \{0\}$ *for all i and j, $0 \leq i, j \leq [k/2]$.*

LEMMA 11. $[L_0, S] \neq \{0\}$.

PROOF. By Lemma 8, we have that $L_{-1} = [L_0, S] + [L_0, T]$. If $[L_0, S] = \{0\}$, we would have by Lemma 4 that $[L_{-1}, L_{-1}] \subseteq [T, T] = \{0\}$, contrary to Lemma 1. Q.E.D.

LEMMA 12. $\mathrm{Ann}_{[L_2, S]} T = \{0\}$.

PROOF. By Lemma 9, $\mathrm{Ann}_{[L_2, S]} T \subseteq \mathrm{Ann}_{L_1} T = [L_2, T]$. By Lemma 8, then, we have that $\mathrm{Ann}_{[L_2, S]} T \subseteq [L_2, T] \cap [L_2, S] = \{0\}$. Q.E.D.

LEMMA 13. $[L_2, S] \neq \{0\}$.

PROOF. If $[L_2, S] = \{0\}$, then by Lemma 8, $L_1 = [L_2, T]$. Hence, we have by Lemma 1 that $L_{-1} = [L_{-2}, L_1] = [L_{-2}, [L_2, T]] = [[L_{-2}, L_2], T] \subseteq T$, so that $\{0\} \neq [L_{-1}, L_{-1}] \subseteq [T, T]$, contrary to Lemma 7. Q.E.D.

LEMMA 14. $\mathrm{Ann}_T [L_2, S] = \{0\}$.

PROOF. Since T is an irreducible L_0-module by Lemma 7, it follows that if $\mathrm{Ann}_T [L_2, S] \neq \{0\}$, then $\mathrm{Ann}_T [L_2, S] = T$. Hence, we would have by Lemma 5 that $[L_{-1}, [L_2, S]] = [S \oplus T, [L_2, S]] \subseteq [S, [L_2, S]] + [T, [L_2, S]] = \{0\}$. Then Lemma 1 yields that $[L_2, S] = \{0\}$, contrary to Lemma 13. Q.E.D.

LEMMA 15. *There exist elements c and q of the center of $[[L_2, S], T]$ and the center of $[[L_2, T], S]$, respectively, such that $\mathrm{Ann}_{[[L_2, S], T]} [L_2, S] = \mathrm{Ann}_{[[L_2, S], T]} T \subseteq \langle c \rangle$ and $\mathrm{Ann}_{[[L_2, T], S]} [L_2, T] = \mathrm{Ann}_{[[L_2, T], S]} S \subseteq \langle q \rangle$.*

PROOF. Set $B = \operatorname{Ann}_{[[L_2,S],T]}[L_2,S]$. If $[B,T] \neq \{0\}$, then by Lemma 7, $[B,T] = T$, so $[[L_2,S],T] = [[L_2,S],[B,T]] = [B,[[L_2,S],T]] \subseteq B$; hence, $[[L_2,S],[[L_2,S],T]] = \{0\}$. Let y be any element of $[L_2,S]$. Then $[T,y]$ is an $[[L_2,S],T]$-submodule (i.e., an ideal) of $[[L_2,S],T]$. Because y is fixed, each element of $[T,y]$ is of the form $[t,y]$, where $t \in T$. Thus, let t be any element of T. We have by Lemma 7 that $\{0\} = (\operatorname{ad} y)^2(\operatorname{ad} t)^2[T,y] = 2(\operatorname{ad}[y,t])^2[T,y]$. By Engel's Theorem, then, $[T,y]$ is nilpotent. But $[T,y]$ is an ideal of $[[L_2,S],T]$, so $[T,y]$ is an ideal of the reductive L_0, and, hence, must be in the center of L_0, as must all of $[[L_2,S],T]$, because y was an arbitrary element of $[L_2,S]$. Consequently, we have by Schur's Lemma that each element x of $[[L_2,S],T]$ acts as a scalar $\xi(x)$ on T, which is irreducible by Lemma 7. As before, let t be any element of T. By Lemma 14, there exists a $z \in [L_2,S]$ such that $[t,z] \neq 0$. For any $x \in [[L_2,S],T]$, we have $0 = [x,[t,z]] = [[x,t],z] = \xi(x)[t,z]$. Thus, $\xi(x) = 0$ for all $x \in [[L_2,S],T]$, so that $\{0\} = [[[L_2,S],T],T] = [B,T]$.

Now consider $C = \operatorname{Ann}_{[[L_2,S],T]} T$. If $[C,[L_2,S]] \neq \{0\}$, then $[T,[C,[L_2,S]]] \neq \{0\}$, since $\operatorname{Ann}_{[L_2,S]} T = \{0\}$ by Lemma 12. Hence, we have $\{0\} \neq [T,[C,[L_2,S]]] = [C,[T,[L_2,S]]] = [C,C]$, the semisimple part of C, so that $[T,[T,[C,[L_2,S]]]] = \{0\}$. Arguing as before, we have by Lemma 10 that if $x \in T$ and $y \in [C,[L_2,S]]$ then $\{0\} = (\operatorname{ad} x)^2(\operatorname{ad} y)^2[T,[C,[L_2,S]]] = 2(\operatorname{ad}[x,y])^2[T,[C,[L_2,S]]]$ so that $[T,[C,[L_2,S]]]$ is central in $[[L_2,S],T]$, contrary to our conclusion above that it must be nonzero and semisimple. Thus, we must have that $[C,[L_2,S]] = \{0\}$, so that $C = B$.

Suppose that B is nonzero, and let $c \in B\backslash\{0\}$. By Lemmas 1 and 7, $\{0\} \neq [c,L_{-1}] = [c,S]$. Since c must be in the center of the reductive L_0, and since S is an irreducible L_0-module by Lemma 7, we have by Schur's Lemma that for all $s \in S$, $[s,c] = \gamma s$ for some nonzero scalar γ. If d is any other nonzero element of B, then of course $[s,d] = \delta s$, $\delta \neq 0$, for all $s \in S$. Then if $v = s + t\,(s \in S, t \in T)$ is any element of L_{-1}, we have that $[v, c/\gamma - d/\delta] = s - s = 0$, so by Lemma 1, $c/\gamma - d/\delta = \{0\}$, and $B = \langle c\rangle$. The rest of the lemma follows by symmetry. Q.E.D.

LEMMA 16. *If no nonzero element of the center of L_0 annihilates L_{-2}, then either $c = 0$ or $q = 0$, where c and q are as in Lemma* 15.

PROOF. If both c and q are nonzero, then there exist nonzero scalars α and β such that $[w,c] = \alpha w$ and $[w,q] = \beta w$ for all $w \in L_{-2}$, which is an irreducible L_0-module by Lemma 1. Then $[L_{-2}, c/\alpha - q/\beta] = \{0\}$, so $c/\alpha = q/\beta$; hence c annihilates both S and T, and we have that $[L_{-1},c] = \{0\}$, contrary to Lemma 1. Q.E.D.

LEMMA 17. *Suppose no nonzero element of the center of L_0 annihilates L_{-2}. Let C be as in the proof of Lemma* 15. *If k is even, we can assume that $C = \{0\}$. If k is odd, we have that $L_0 = [[L_2,S],T]$.*

PROOF. If k is even, then S and T are symmetric, so by relabelling if necessary, we can assume by Lemmas 15 and 16 that $C = B = \{0\}$.

If k is odd, then $[S,L_k] = \{0\}$ (see (2)ff). Then by Lemmas 6, 7, 8, and 9 $L_k(\operatorname{ad} L_{-2})^{(k+1)/2} = S$, so $[[L_2,T],S] \subseteq [L_k,[[L_2,T],L_{-2}]](\operatorname{ad} L_{-2})^{(k-1)/2} = [L_k,T](\operatorname{ad} L_{-2})^{(k-1)/2} = [L_k(\operatorname{ad} L_{-2})^{(k-1)/2},T] \subseteq [[L_2,S],T]$. Thus, by Lemmas

2, 7, and 5,

$$L_0 = [[L_2, L_{-1}], L_{-1}] = [[L_2, S], T] + [[L_2, T], S] \subseteq [[L_2, S], T],$$

so that $L_0 = [[L_2, S], T]$. Q.E.D.

Set $N = T \oplus [[L_2, S], T] \oplus [L_2, S]$. Then N is a graded Lie algebra, where for all i, $-1 \leq i \leq 1$, the ith gradation space of N is $N \cap L_i$.

LEMMA 18. *$N = T \oplus [[L_2, S], T] \oplus [L_2, S]$ is an irreducible graded Lie algebra whenever L_0 contains no nonzero central element which annihilates L_{-2}.*

PROOF. Let Q be a nonzero $[[L_2, S], T]$-submodule of T. Then $[[L_2, S], Q]$ is an $[[L_2, S], T]$-submodule of $[[L_2, S], T]$. Furthermore, since L_0 is reductive, $L_0 = [[L_2, S], T] \oplus R$ for some ideal R of L_0 such that $[R, [[L_2, S], T]] = \{0\}$. Hence, $[[L_2, S], Q]$ is an L_0-submodule of $[[L_2, S], T]$, so $[[[L_2, S], Q], T]$ is an L_0-submodule of T. By Lemma 7, $[[[L_2, S], Q], T]$ cannot be a proper submodule of T. If $\{0\} = [[[L_2, S], Q], T] = [[[L_2, S], T], Q]$, then $Q \subseteq \mathrm{Ann}_T[[L_2, S], T] = T$, since $\mathrm{Ann}_T[[L_2, S], T]$ is an L_0-submodule of the irreducible L_0-submodule T. Let C be as above in the proof of Lemma 15. We have that $[[L_2, S], T] \subseteq C$.

If k is odd, we have by Lemma 17 that $L_0 = [[L_2, S], T] \subseteq C$, so that $[L_0, T] = \{0\}$, contrary to Lemma 11. If k is even, we have by Lemma 17 that $C = \{0\}$. But then we would have, for example, that $\mathrm{Ann}_{[L_2, S]} T = [L_2, S] \neq \{0\}$ by Lemma 13, contrary to Lemma 12. Hence, we must have that $[[[L_2, S], Q], T] = T$, and we have

$$Q \supseteq [[[L_2, S], T], Q] = [[[L_2, S], Q], T] = T. \qquad \text{Q.E.D.}$$

LEMMA 19. *For all i, $-2 \leq i \leq k-1$, L_i is spanned by weight vectors of the form $[x, y]$, where x is a weight vector of L_{-1} and y is a weight vector of L_{i+1}. L_k is spanned by weight vectors, also.*

PROOF. S, T, and L_k are irreducible L_0-modules by Lemmas 2 and 7. Hence, each is equal to the set of all weight vectors which it contains, since each of these sets is an L_0-submodule which is nonempty due to the fact that each of the L_0-modules S, T, and L_k is a finite-dimensional vector space over an algebraically closed field, and the Cartan subalgebra of L_0 is commutative. Lemma 19 now follows from Lemma 2. Q.E.D.

PROOF OF THEOREM. By Lemma 3, and our assumption that the center of L_0 contains no nonzero element which annihilates L_{-2}, we can apply the Theorem of [**2**] to show that L possesses a gradation of the form (2), where $M_0 = T \oplus L_0 \oplus [L_2, S]$. If c is as in Lemma 15 then by Lemmas 12, 14, and 16, the Lie algebra $N/\langle c\rangle$ is a transitive graded Lie algebra. By Lemma 18, N is an irreducible graded Lie algebra. In view of Lemmas 12 and 19 we can use the arguments of Lemma 8 and §4 of [**3**] to show that the representation of $[[L_2, S], T]/\langle c\rangle$ in T is restricted. Then the hypotheses of Theorem 3 of [**4**] are satisfied, so $N/\langle c\rangle$ must be classical and reductive. Thus, if k is even, we have from Lemma 17 that $M_0 = N + L_0$ is classical and reductive. If k is odd, then by Lemma 17 again, $M_0 = N + L_0 \subseteq N$ modulo its center (which by Lemmas 15 and 16 is of dimension at most one) is simple. In view of (for example) the paragraph following Theorem 3 of [**4**], we can now use the proof of the Theorem of [**3**] to conclude that L is classical or of Cartan type.

REFERENCES

1. T. B. Gregory, *A characterization of the contact Lie algebras*, Proc. Amer. Math. Soc. **82** (1981), 505–511.
2. ——, *On simple reducible Lie algebras of depth two*, Proc. Amer. Math. Soc. **83** (1981), 31–35.
3. ——, *Simple Lie algebras with classical reductive null component*, J. Algebra **63** (1980), 484–493.
4. V. G. Kac, *The classification of the simple Lie algebras over a field with non-zero characteristic*, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 385–408; English transl., Math. USSR Izv. **4** (1970), 391–413.

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