ON SIMPLE REDUCIBLE DEPTH-TWO LIE ALGEBRAS
WITH CLASSICAL REDUCTIVE NULL COMPONENT

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ABSTRACT. We classify the simple finite-dimensional reducible graded Lie algebras of the form \( L = L_2 \oplus L_1 \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_k \) over an algebraically closed field of characteristic greater than 3, where \( L_0 \) is reductive and classical such that no nonzero element of the center of \( L_0 \) annihilates \( L_2 \) and where \( L_1 \) is the sum of two proper \( L_0 \)-submodules.

In [2], the present author showed that if the null component of a simple finite-dimensional Lie algebra of the form

\[
L = L_2 \oplus L_1 \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_k
\]

contains no nonzero abelian ideal which annihilates \( L_2 \), and if \( L_1 = S + T \), where \( S \) and \( T \) are proper \( L_0 \)-submodules of \( L_1 \), then \( S \) and \( T \) are abelian irreducible \( L_0 \)-submodules, and \( L_1 = S \oplus T \). In addition, \( L \) possesses an irreducible transitive gradation of the form

\[
L = M_1 \oplus M_0 \oplus \cdots \oplus M_{[k/2]},
\]

such that (by interchanging the names of \( S \) and \( T \) if necessary) \( M_0 = T \oplus L_0 \oplus [L_2, S] \) and \( [L_2, S] = \{0\} \) if \( k \) is odd.

In this paper, we strengthen the hypotheses on \( L_0 \); we assume that \( L_0 \) is classical and reductive and that \( \text{Ann}_{L_0} L_2 \) contains no nonzero element of the center of \( L_0 \). We then prove that \( L \) is either classical or of Cartan type. Specifically, we prove the following

THEOREM. Let \( L \) be a simple finite-dimensional reducible graded Lie algebra of the form (1) over an algebraically closed field of characteristic greater than 3, and suppose that \( L_0 \) is a classical reductive Lie algebra no nonzero element of whose center annihilates \( L_2 \), and that \( L_1 = S + T \), where \( S \) and \( T \) are proper \( L_0 \)-submodules of \( L_1 \). Then \( L \) is classical or of Cartan type.

(For definitions of the terms "transitive" and "irreducible" as they are used in this paper, see [1].)

The proof of this theorem will be carried out by means of a series of lemmas. We begin by recalling certain of the lemmas of [2] for which we will have use here. The numbers in parentheses after the first 10 lemma numbers are those by which the lemmas were designated in [2]. Proofs of these lemmas can be found in [2] (or [1]). In the first 4 lemmas, we assume only that \( L \) is a simple graded Lie algebra of the form (1).
LEMMA 1. \( L_{-2} \) is an irreducible \( L_0 \)-module. Furthermore, \( [L_{-1}, L_{-1}] \neq \{0\} \), so \( [L_{-1}, L_{-1}] = L_{-2} \); also, \( L_{-1} = [L_{-2}, L_1] \). Lastly, \( [L_{-1}, x] \neq \{0\} \) for all \( x \notin L_{-2} \).

LEMMA 2. \( L_k \) is an irreducible \( L_0 \)-module, and \( L_{j-1} = [L_j, L_{-1}] \) for all \( j, -1 \leq j \leq k \).

LEMMA 3. We have that \([L_0, L_k] \neq \{0\}\) and that \([L_{-2}, L_i] \neq \{0\}\) for all \( i, 0 \leq i \leq k \).

LEMMA 4. \( L_{-1} \) contains no proper nonabelian \( L_0 \)-submodule.

LEMMA 5. \([S, [S, L_2]] = \{0\}\).

LEMMA 6. \( \text{Ann}_{L_i} L_{-2} = \{0\} \) for all \( i, 1 \leq i \leq k \).

LEMMA 7. \( L_{-1} = S \oplus T \), where \( S \) and \( T \) are irreducible abelian \( L_0 \)-submodules of \( L_{-1} \).

LEMMA 8. \( L_{2i-1} = [L_{2i}, S] \oplus [L_{2i}, T] \) for all \( i, 0 \leq i \leq [k/2] \).

LEMMA 9. \( \text{Ann}_{L_{2i-1}} S = [L_{2i}, S] \) for all \( i, 0 \leq i \leq [k/2] \).

LEMMA 10. \([L_{2i}, S], [L_{2j}, S] = \{0\}\) for all \( i, j, 0 \leq i, j \leq [k/2] \).

LEMMA 11. \([L_0, S] \neq \{0\}\).

PROOF. By Lemma 8, we have that \( L_{-1} = [L_0, S] + [L_0, T] \). If \([L_0, S] = \{0\}\), we would have by Lemma 4 that \([L_{-1}, L_{-1}] \subseteq \{T, T\} = \{0\}\), contrary to Lemma 1. Q.E.D.

LEMMA 12. \( \text{Ann}_{[L_2, S]} T = \{0\} \).

PROOF. By Lemma 9, \( \text{Ann}_{[L_2, S]} T \subseteq \text{Ann}_{L_1} T = [L_2, T] \). By Lemma 8, then, we have that \( \text{Ann}_{[L_2, S]} T \subseteq [L_2, T] \cap [L_2, S] = \{0\} \). Q.E.D.

LEMMA 13. \([L_2, S] \neq \{0\}\).

PROOF. If \([L_2, S] = \{0\}\), then by Lemma 8, \( L_1 = [L_2, T] \). Hence, we have by Lemma 1 that \( L_{-1} = [L_{-2}, L_1] = [L_{-2}, [L_2, T]] = [[L_{-2}, L_2], T] \subseteq T \), so that \( \{0\} \neq [L_{-1}, L_{-1}] \subseteq [T, T], \) contrary to Lemma 7. Q.E.D.

LEMMA 14. \( \text{Ann}_T [L_2, S] = \{0\} \).

PROOF. Since \( T \) is an irreducible \( L_0 \)-module by Lemma 7, it follows that if \( \text{Ann}_T [L_2, S] \neq \{0\} \), then \( \text{Ann}_T [L_2, S] = T \). Hence, we would have by Lemma 5 that \([L_{-1}, L_2, S] = [S \oplus T, [L_2, S]] \subseteq [S, [L_2, S]] + [T, [L_2, S]] = \{0\} \). Then Lemma 1 yields that \([L_2, S] = \{0\}\), contrary to Lemma 13. Q.E.D.

LEMMA 15. There exist elements \( c \) and \( q \) of the center of \([L_2, S], T\) and the center of \([L_2, T], S\), respectively, such that \( \text{Ann}_{[[L_2, S], T]} [L_2, S] = \text{Ann}_{[[L_2, S], T]} T \subseteq \langle c \rangle \) and \( \text{Ann}_{[[L_2, T], S]} [L_2, T] = \text{Ann}_{[[L_2, T], S]} S \subseteq \langle q \rangle \).
PROOF. Set $B = \text{Ann}_{[[L_2,S],T]}[L_2,S]$. If $[B,T] \neq \{0\}$, then by Lemma 7, $[B,T] = T$, so $[[L_2,S],T] = [[L_2,S],[B,T]] = [B,[[L_2,S],T]] \subseteq B$; hence, $[[L_2,S],[L_2,S],T]] = \{0\}$. Let $y$ be any element of $[L_2,S]$. Then $[T,y]$ is an $[[L_2,S],T]$-submodule (i.e., an ideal) of $[[L_2,S],T]$. Because $y$ is fixed, each element of $[T,y]$ is of the form $[t,y]$, where $t \in T$. Thus, let $t$ be any element of $T$. We have by Lemma 7 that $\{0\} = (ad y)^2(ad t)^2[T,y] = 2(ad[y,t])^2[T,y]$. By Engel's Theorem, then, $[T,y]$ is nilpotent. But $[T,y]$ is an ideal of $[[L_2,S],T]$, so $[T,y]$ is an ideal of the reductive $L_0$, and, hence, must be in the center of $L_0$, as must all of $[[L_2,S],T]$, because $y$ was an arbitrary element of $[L_2,S]$. Consequently, we have by Schur's Lemma that each element $x$ of $[[L_2,S],T]$ acts as a scalar $\xi(x)$ on $T$, which is irreducible by Lemma 7. As before, let $t$ be any element of $T$. By Lemma 14, there exists a $z \in [L_2,S]$ such that $[t,z] \neq 0$. For any $x \in [[L_2,S],T]$, we have $0 = [x,[t,z]] = [[x,t],z] = \xi(x)[t,z]$. Thus, $\xi(x) = 0$ for all $x \in [[L_2,S],T]$, so that $\{0\} = [[[L_2,S],T],T] = [B,T]$.

Now consider $C = \text{Ann}_{[[L_2,S],T]} T$. If $[C,[L_2,S]] \neq \{0\}$, then $[T,[C,[L_2,S]]] \neq \{0\}$, since $\text{Ann}_{[[L_2,S],T]} T = \{0\}$ by Lemma 12. Hence, we have $\{0\} \neq [T,[C,[L_2,S]]] = [C,[T,[L_2,S]]] = [C,C]$, the semisimple part of $C$, so that $[T,[T,[C,[L_2,S]]]] = \{0\}$. Arguing as before, we have by Lemma 10 that if $x \in T$ and $y \in [C,[L_2,S]]$ then $\{0\} = (ad x)^2(ad y)^2[T,[C,[L_2,S]]] = 2(ad[x,y])^2[T,[C,[L_2,S]]]$ so that $[T,[C,[L_2,S]]]$ is central in $[[L_2,S],T]$, contrary to our conclusion above that it must be nonzero and semisimple. Thus, we must have that $[C,[L_2,S]] = \{0\}$, so that $C = B$.

Suppose that $B$ is nonzero, and let $c \in B \setminus \{0\}$. By Lemmas 1 and 7, $\{0\} \neq [c,L_{-1}] = [c,S]$. Since $c$ must be in the center of the reductive $L_0$, and since $S$ is an irreducible $L_0$-module by Lemma 7, we have by Schur's Lemma that for all $s \in S$, $[s,c] = \gamma s$ for some nonzero scalar $\gamma$. If $d$ is any other nonzero element of $B$, then of course $[s,d] = \delta s$, $\delta \neq 0$, for all $s \in S$. Then if $v = s + t$ ($s \in S, t \in T$) is any element of $L_{-1}$, we have that $[v,c/\gamma - d/\delta] = s - s = 0$, so by Lemma 1, $c/\gamma - d/\delta = \{0\}$, and $B = \langle c \rangle$. The rest of the lemma follows by symmetry. Q.E.D.

**Lemma 16.** If no nonzero element of the center of $L_0$ annihilates $L_{-2}$, then either $c = 0$ or $q = 0$, where $c$ and $q$ are as in Lemma 15.

**Proof.** If both $c$ and $q$ are nonzero, then there exist nonzero scalars $\alpha$ and $\beta$ such that $[w,c] = \alpha w$ and $[w,q] = \beta w$ for all $w \in L_{-2}$, which is an irreducible $L_0$-module by Lemma 1. Then $[L_{-2},c/\alpha - q/\beta] = \{0\}$, so $c/\alpha = q/\beta$; hence $c$ annihilates both $S$ and $T$, and we have that $[L_{-1},c] = \{0\}$, contrary to Lemma 1. Q.E.D.

**Lemma 17.** Suppose no nonzero element of the center of $L_0$ annihilates $L_{-2}$. Let $C$ be as in the proof of Lemma 15. If $k$ is even, we can assume that $C = \{0\}$. If $k$ is odd, we have that $L_0 = [[L_2,S],T]$.

**Proof.** If $k$ is even, then $S$ and $T$ are symmetric, so by relabelling if necessary, we can assume by Lemmas 15 and 16 that $C = B = \{0\}$.

If $k$ is odd, then $[S,L_k] = \{0\}$ (see (2)ff). Then by Lemmas 6, 7, 8, and 9 $L_k(ad L_{-2})^{(k+1)/2} = S$, so $[[L_2,T],S] \subseteq [L_k,[[L_2,T],L_{-2}]](ad L_{-2})^{(k-1)/2} = [L_k,T](ad L_{-2})^{(k-1)/2}$, $[L_2,T] \subseteq [[L_2,S],T]$. Thus, by Lemmas
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2, 7, and 5,

\[ L_0 = [[L_2, L_{-1}], L_{-1}] = [[L_2, S], T] + [[L_2, T], S] \subseteq [[L_2, S], T], \]

so that \( L_0 = [[L_2, S], T] \). Q.E.D.

Set \( N = T \oplus [[L_2, S], T] \oplus [L_2, S] \). Then \( N \) is a graded Lie algebra, where for all \( i, -1 \leq i \leq 1 \), the \( i \)th gradation space of \( N \) is \( N \cap L_i \).

**Lemma 18.** \( N = T \oplus [[L_2, S], T] \oplus [L_2, S] \) is an irreducible graded Lie algebra whenever \( L_0 \) contains no nonzero central element which annihilates \( L_{-2} \).

**Proof.** Let \( Q \) be a nonzero \([L_2, S], T\)-submodule of \( T \). Then \([L_2, S], Q\) is a \([L_2, S], T\)-submodule of \([L_2, S], T\). Furthermore, since \( L_0 \) is reductive, \( L_0 = [[L_2, S], T] \oplus R \) for some ideal \( R \) of \( L_0 \) such that \([R, [[L_2, S], T]] = \{0\}\).

Hence, \([L_2, S], Q\) is an \( L_0\)-submodule of \([L_2, S], T\), so \([[[L_2, S], Q], T] \) is an \( L_0\)-submodule of \( T \). By Lemma 7, \([[[L_2, S], Q], T] \) cannot be a proper submodule of \( T \).

If \( \{0\} = [[[L_2, S], Q], T] = [[[L_2, S], T], Q], \) then \( Q \subseteq \text{Ann}_T[[L_2, S], T] = T, \) since \( \text{Ann}_T[[L_2, S], T] \) is an \( L_0\)-submodule of the irreducible \( L_0\)-submodule \( T \). Let \( C \) be as above in the proof of Lemma 15. We have that \([[[L_2, S], T] \subseteq C. \)

If \( k \) is odd, we have by Lemma 17 that \( L_0 = [[L_2, S], T] \subseteq C, \) so that \([L_0, T] = \{0\}, \) contrary to Lemma 11. If \( k \) is even, we have by Lemma 17 that \( C = \{0\}. \) But then we would have, for example, that \( \text{Ann}_{[L_2, S]} T = [L_2, S] \neq \{0\} \) by Lemma 13, contrary to Lemma 12. Hence, we must have that \([[[L_2, S], Q], T] = T, \) and we have

\[ Q \supset [[[L_2, S], T], Q] = [[[L_2, S], Q], T] = T. \] Q.E.D.

**Lemma 19.** For all \( i, -2 \leq i \leq k - 1 \), \( L_i \) is spanned by weight vectors of the form \([x, y]\), where \( x \) is a weight vector of \( L_{-1} \) and \( y \) is a weight vector of \( L_{i+1}. \) \( L_k \) is spanned by weight vectors, also.

**Proof.** \( S, T, \) and \( L_k \) are irreducible \( L_0\)-modules by Lemmas 2 and 7. Hence, each is equal to the set of all weight vectors which it contains, since each of these sets is an \( L_0\)-submodule which is nonempty due to the fact that each of the \( L_0\)-modules \( S, T, \) and \( L_k \) is a finite-dimensional vector space over an algebraically closed field, and the Cartan subalgebra of \( L_0 \) is commutative. Lemma 19 now follows from Lemma 2. Q.E.D.

**Proof of Theorem.** By Lemma 3, and our assumption that the center of \( L_0 \) contains no nonzero element which annihilates \( L_{-2} \), we can apply the Theorem of [2] to show that \( L \) possesses a gradation of the form (2), where \( M_0 = T \oplus L_0 \oplus [L_2, S]. \) If \( c \) is as in Lemma 15 then by Lemmas 12, 14, and 16, the Lie algebra \( N/(c) \) is a transitive graded Lie algebra. By Lemma 18, \( N \) is an irreducible graded Lie algebra. In view of Lemmas 12 and 19 we can use the arguments of Lemma 8 and \S 4 of [3] to show that the representation of \([[[L_2, S], T]/(c) \) in \( T \) is restricted. Then the hypotheses of Theorem 3 of [4] are satisfied, so \( N/(c) \) must be classical and reductive. Thus, if \( k \) is even, we have from Lemma 17 that \( M_0 = N + L_0 \) is classical and reductive. If \( k \) is odd, then by Lemma 17 again, \( M_0 = N + L_0 \subseteq N \) modulo its center (which by Lemmas 15 and 16 is of dimension at most one) is simple. In view of (for example) the paragraph following Theorem 3 of [4], we can now use the proof of the Theorem of [3] to conclude that \( L \) is classical or of Cartan type.
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References


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