PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 3, July 1982

A REMARK CONCERNING MULTIPLICITIES

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ABSTRACT. We prove that if a complete local ring A containing a field satisfies Serre's condition S_n and the multiplicity of A is at most n, then A must be Cohen-Macaulay.

The purpose of this paper is to prove the following theorem.

THEOREM 1.1. *Let (A, m) be a complete local ring containing a field. Suppose A satisfies Serre's condition S_n, and let $e(A)$ be the multiplicity of A. If $e(A) \leq n$, then A is Cohen-Macaulay.*

Recall a Noetherian ring A is said to satisfy S_n if

$$\operatorname{depth} A_p \geq \inf(n, \operatorname{ht}(p)) \quad \text{for all } p \in \operatorname{Spec}(A).$$

Two corollaries of Theorem 1.1 are known.

COROLLARY 1.2 [4]. *Let (A, m) be a complete local ring containing a field which is unmixed and has multiplicity one. Then A is regular.*

PROOF. We may assume A/m is infinite. Choose a minimal reduction [5] $x_1, \ldots, x_d$ for m, where $d = \dim(A)$. Then $e(A) = e(\mathbf{x}; A)$, the multiplicity of A with respect to $\mathbf{x} = (x_1, \ldots, x_d)$ [5, Theorem 1, p. 46]. Since A is Cohen-Macaulay by Theorem 1.1, $e(\mathbf{x}; A) = l(A/\mathbf{x})$, the length of $A/\mathbf{x}$ [6]. However, since $e(A) = 1$, $\mathbf{x}$ must be equal to m. Thus A is regular.

COROLLARY 1.3 (IKEDA). *Let (A, m) be a complete local ring containing a field such that A satisfies S_2, and $e(A) = 2$. Then A is a hypersurface; that is, $A = R/(f)$ where R is a regular local ring and $f \neq 0$.*

PROOF. As in Corollary 1.2, we may assume $k = A/m$ is infinite; and $\mathbf{x} = (x_1, \ldots, x_d)$ is a minimal reduction of m. Thus, $2 = e(A) = e(\mathbf{x}; A) = l(A/\mathbf{x})$, the latter equality following from the fact that A is necessarily Cohen-Macaulay by Theorem 1.1. Therefore $l(m/\mathbf{x}) = 1$, and we may map $R = k[[X_1, \ldots, X_d, Y]]$ onto A by sending X_i to x_i and Y to the lifting of a generator of $m/\mathbf{x}$. Since $\dim A = d$, the kernel of the map from R onto A is a height one ideal I of R. Since A is Cohen-Macaulay, I is unmixed. As R is factorial $I = (f)$ is principal.

REMARK 1. Both corollaries have been shown without the assumption that A contains a field, and most probably Theorem 1.1 is valid in this case. However we shall use the direct summand theorem and the syzygy theorem, which are only known to hold when A contains a field.

Received by the editors November 11, 1981.
1980 *Mathematics Subject Classification*. Primary 13H15.
Key words and phrases. Multiplicity, local ring, Cohen-Macaulay.
[1]Supported by NSF Postdoctoral Fellowship.

REMARK 2. Nagata [**4**, Example 2, p. 203] gives an example of a local domain of multiplicity one which is not regular. Thus the completeness of A is vital. Of course, this assumption may be removed if we assume A has Cohen-Macaulay formal fibers.

PROOF OF THEOREM 1.1. As above, we may assume $k = A/m$ is infinite and choose a minimal reduction $\mathbf{x} = (x_1, \ldots, x_d)$ of m. Set $R = k[[x_1, \ldots, x_d]] \subseteq A$. Then A is a finite R-module, and R is isomorphic to a power-series ring in d-variables over k. In particular, R is regular. We have the equality $e(A) = e(\mathbf{x}; A)$, and $e(\mathbf{x}; A)$ is just the multiplicity $e_R(A)$ of A as an R-module.

Observe that $e_R(A) = \operatorname{rank}_R A$. By definition, $\operatorname{rank}_R A = \dim_K K \otimes_R A$, where K is the fraction field of R. This equality can be seen as follows. If M is an R-module the degree of the Hilbert polynomial of M is bounded by $\dim(M)$. Consequently if $0 \to N \to M \to T \to 0$ is exact and $\dim T < \dim N = \dim M$, then it follows that $e_R(N) = e_R(M)$. It now easily follows that if M is a module of positive rank k over R, then $e_R(M) = k$.

By the direct summand theorem [**3**, Theorem 2], the inclusion $R \subseteq A$ splits as R-module. Write $A = R \bigoplus M$. (Here we use the fact that A contains a field.) Since $\operatorname{rank} A = e(A) \leq n$, $\operatorname{rank} M \leq n - 1$.

Recall that a module N over R is said to satisfy S_k if $\operatorname{depth} N_p \geq \min(k, \operatorname{ht} p)$ for all $p \in \operatorname{Spec}(R)$. We claim M satisfies S_n. Let $p \in \operatorname{Spec}(R)$. Then $\operatorname{depth} A_p = \min(\operatorname{depth} M_p, \operatorname{depth} R_p)$, so that $\operatorname{depth} M_p \geq \operatorname{depth} A_p$.

As A is finite over R, $\operatorname{ht}(pA) = \operatorname{ht} p$. It follows that $\operatorname{depth} A_p \geq \min(n, \operatorname{ht} p)$ since A satisfies S_n. Thus M satisfies S_n. By the theorem of Auslander and Bridger [**1**], M is an nth syzygy. We may now use the recent result of Evans and Griffith [**2**]: if M is a module of finite projective dimension, and M is an nth syzygy, then M is free provided $\operatorname{rank} M < n$. As $\operatorname{rank} M \leq n - 1$ in our case, we conclude that M (and thus A) is a free R-module. It follows that A is Cohen-Macaulay.

REMARKS AND ACKNOWLEDGEMENT. The author would like to point out that the idea of using the direct summand conjecture is due to Ikeda in his proof of Corollary 1.2, and also would like to thank S. Goto for valuable comments. Finally we note that Hochster has shown that the direct summand conjecture implies the syzygy conjecture.

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