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COMPLETION OF A CONSTRUCTION OF JOHNSTONE

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ABSTRACT. A complete lattice is constructed which is not sober in the Scott topology.

Peter Johnstone has constructed [**3**] a (countable) partially ordered set X^+ which admits a sober topology (i.e. the relation $x \in \{y\}^-$ gives the partial order) but is not sober in the Scott topology. He asked whether such an example can be a complete lattice; this note shows that it can.

Some of the background should be mentioned, as it is in [**3**]. Every complete lattice admits a sober topology; but the topology that R.-E. Hoffmann used for this [**2**] is a bit tricky to deal with. There is a condition for sobriety of the Scott topology on a complete lattice [**1**, II, 4.14] which Johnstone calls "mild".

The definition of the Scott topology of a partially ordered set is just this: a set is closed provided it contains all predecessors of its elements and suprema of its nonempty up-directed subsets.

The following construction is rather closely based on Johnstone's; but, to make it work in a complete lattice, I need uncountably many coatoms. Let ω be the set of natural numbers, $\omega + 1 = \omega \cup \{\omega\}$, and A the set of (2^ω) ordinals less than 2^ω. Let $B = \omega \times (\omega + 1)$, $C = \omega^\omega \times \omega$. Order B by $(m, n) \leq (m', n')$ if (1) $m = m'$ and $n \leq n'$, or (2) $n' = \omega$ and $m < m'$. Order C by $(f, n) \leq (f', n')$ if $f = f'$ and $n \leq n'$. Let $I \subset A^2 \times B \times C$ be the set of all ordered quadruples $(\alpha, \beta, (m, n), (f, k))$ for which $\alpha \neq \beta$, $n = f(m)$ (thus $n \neq \omega$), and $k = m$. The power of I is only 2^ω, so there is an injection $i\colon I \to A$. We can require, too, that $i(\alpha, \beta, (m, f(m)), (f, m)) > \alpha \vee \beta$; and we call it $i(\alpha, \beta, f, m)$ for short.

Let S consist of $B \cup C$ with a greatest element T added (elements of B being incomparable with elements of C). Let $X = A \times S$. Generate a partial order on X (by transitive closure) from (1) the order of S in each fiber $\{\alpha\} \times S$, and (2) the relations $(i(\alpha, \beta, f, m), T) \geq (\alpha, (m, f(m)))$ and $(i(\alpha, \beta, f, m), T) \geq (\beta, (f, m))$.

Observe that up-directed subsets of X have suprema. Our complete lattice L consists of the intersections of principal order ideals in X. Most of what we have to check is that the nonprincipal ideals in L are "sparse" and do not affect the up-directed sets. First, the principal ideals are the sequences $\{(\alpha, (m, x))\colon x \leq n\}$, with n finite or $m = 0$, the unions of $m + 1$ sequences $\{(\alpha, (x, y))\colon x \leq m\}$, the finite sequences $\{(\alpha, (f, z))\colon z \leq n\}$, the sets $\{\alpha\} \times S$ with α not a value of i, and one more type: for $\gamma = i(\alpha, \beta, f, m)$, the predecessors of (γ, T) form $\{\gamma\} \times S$, a finite sequence under $(\alpha, (m, f(m)))$, and a finite sequence under $(\beta, (f, m))$. The intersections include of course $\emptyset$ and X. We shall find the remaining ones, the *nontrivial* intersections. Observe that an intersection contained in one fiber $\{\alpha\} \times S$

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is trivial. So we need to look at intersections of (first) pairs of principal ideals of the last type, under $(\gamma, T) = (i(\alpha, \beta, f, m), T)$ and $(\gamma', T) = (i(\alpha', \beta', f', m'), T)$. If the intersection M is nontrivial then $\{\alpha, \beta, \gamma\}$ and $\{\alpha', \beta', \gamma'\}$ have two common elements—not three, since the largest elements γ, γ' are different. Impossible that $\alpha = \alpha'$, $\beta = \beta'$ (and M nontrivial). For if $(m, f(m))$ and $(m', f'(m'))$ have a common predecessor we have $m = m'$; since $\gamma \neq \gamma'$, $f \neq f'$ and (f, m) has no common predecessor with (f', m). Impossible, of course, that $\alpha = \beta'$ and $\beta = \alpha'$. Choosing notation so that $\gamma' > \gamma$, the possible cases are (1) $\gamma = \alpha'$, $\beta = \beta'$, and (2) $\gamma = \beta'$, $\alpha = \alpha'$. In case (1), a nontrivial intersection M must contain $(\beta, (f, 0))$, so $f' = f$. No third principal ideal J has nontrivial intersection with M. It could only be under $(\gamma'', T) = (i(\alpha'', \beta'', f'', m''), T)$; $\gamma'' \neq \beta$, since then J would not meet $\{\gamma\} \times S$, so $\beta'' = \beta$ and $\alpha'' = \gamma$. Again $f'' = f$; so $m'' \neq m'$ (since $\gamma'' \neq \gamma'$), and $(m'', f(m''))$ has no common predecessor with $(m', f(m'))$, making the intersection trivial. It follows that two nontrivial intersections are incomparable if one of them is in this case.

Similarly, in case (2), common predecessors of $(\alpha, (m, f(m)))$ and $(\alpha, (m', f'(m')))$ require $m' = m$. A third principal ideal under (γ'', T) having nontrivial intersection with M requires $\alpha'' = \alpha = \alpha'$, $\beta'' = \beta'$, $m'' = m'$; so $f'' \neq f'$ and the parts in $\{\gamma\} \times S$ are disjoint. And all nontrivial intersections are incomparable. Also, each has only finitely many predecessors. So embedding X in L by principal ideals preserves suprema of up-directed sets.

Now in the complete lattice L, the complement L^- of the singleton $\{X\}$ certainly is not the Scott closure of a point. But L^- is Scott closed: obviously downward closed, and an up-directed subset has at most one nontrivial element and has its supremum in L^-. Finally, L^- has no disjoint nonempty open sets U_1, U_2. They would contain maximal elements (α_1, T) resp. (α_2, T). Then U_1 must contain almost all of the elements $(\alpha_1, (n, \omega))$ (all for $n \geq n_1$); and for each of those n it must contain all $(\alpha_1, (n, x))$ for $x \geq f(n)$, for some $f \in \omega^\omega$. Now if U_1 and U_2 are disjoint, $\alpha_1 \neq \alpha_2$; and U_2 contains almost all $(\alpha_2, (f, y))$, all for $y \geq n_2$. Pick $n \geq n_1 \vee n_2$, and we have $(\alpha_1, (n, f(n))) \in U_1$, $(\alpha_2, (f, n)) \in U_2$. So both contain $(i(\alpha_1, \alpha_2, f, n), T)$, a contradiction.

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