PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 3, July 1982

COEFFICIENTS AND NORMAL FUNCTIONS

PETER LAPPAN

ABSTRACT. Let $f(z) = \sum a_n z^n$ be an analytic function in the unit disc. It is proved that if $\{a_n\}$ is a bounded monotone sequence of real numbers, or if $\sum |a_n - a_{n-1}| < \infty$ and $a_n \nrightarrow 0$, then $f(z)$ is a normal function. Examples are given to show that these results are delicate.

1. Let $D = \{z\colon |z| < 1\}$, let $C = \{z\colon |z| = 1\}$, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function analytic in D. The function f is said to be a *normal function* if the family of functions $\{f(z\colon a, \theta) = f(e^{i\theta}(z + a)/(1 + \bar{a}z))\colon a \in D, \theta \in [0, 2\pi)\}$ is a normal family in the sense of Montel, that is, each sequence of functions in the family contains a subsequence which converges uniformly on each compact subset of D either to an analytic function or to ∞. (When we say that the sequence $\{f_n(z)\}$ converges uniformly to ∞ on a set S we mean that the sequence $\{g_n(z) = 1/f_n(z)\}$ converges uniformly to the zero function on S.) The function f is called a *Bloch function* if the family $\{f(z\colon a, \theta) - f(a)\colon a \in D, \theta \in [0, 2\pi)\}$ is a normal family in the sense of Montel. Other characterizations of normal functions and Bloch functions are well known (see, for example, [**1** and **4**]).

Coefficients of Bloch functions have been studied by Anderson, Clunie, and Pommerenke [**1**], Mathews [**5**], and Neitzke [**6**], among others. Although the coefficients of normal functions have not been studied comprehensively, some isolated results have appeared. Mathews [**5**] gave an example of a nonnormal function for which the coefficients converge to zero. Sons and Campbell [**7**] have characterized gap series which are normal functions. And Campbell and Piranian [**3**] have given an example of a bounded (and thus normal) analytic function $f(z) = \sum a_n z^n$ for which $g(z) = \sum |a_n| z^n$ is not a normal function.

In the present paper, some conditions on the coefficients which imply that a function is a normal function are given. It is proved that if $\{a_n\}$ is a bounded monotone sequence of real numbers, then $f(z)$ is a normal function. Further, if $\{a_n\}$ is a sequence of complex numbers such that $\sum |a_n - a_{n-1}| < \infty$ and $a_n \nrightarrow 0$, then $f(z)$ is a normal function. These and other results of this general nature dealing with bounded coefficients are presented in §3. In §4, some examples are given which show that the hypotheses of the results in §3 cannot be substantially relaxed. In §5, the analogous situation for unbounded coefficients is considered.

2. Before stating the results and examples, we need to give several criteria for a function to be normal and for a function to be nonnormal.

DEFINITION. *Let $f(z)$ be a function analytic in D. We say that the point $p \in C$ is a nonnormal point for the function f if there exists a sequence of points $\{a_n\}$ in*

Received by the editors June 5, 1981.
1980 *Mathematics Subject Classification.* Primary 30D45.
Key words and phrases. Normal function, nonnormal point.

D such that $a_n \to p$ and the family $\{f(z\colon a_n, \theta)\colon \theta \in [0, 2\pi), n = 1, 2, 3, \dots\}$ is not a normal family.

It follows easily from the definition of a normal function that a nonnormal function f must have at least one nonnormal point for f. We state this in lemma form.

LEMMA 1. *An analytic function f in D is a normal function if and only if there are no points of C which are nonnormal points for f.*

The following lemmas are easily proved by elementary normal family arguments.

LEMMA 2. *If f and g are analytic functions in D, if $p \in C$ is a nonnormal point for f, and if g is bounded in some neighborhood of p, then p is a nonnormal point for the function $f + g$. Further, if g is a bounded analytic function in D, then the point $q \in C$ is a nonnormal point for the function f if and only if q is a nonnormal point for the function $f + g$.*

LEMMA 3. *If f is analytic in D and continuous on $D \cup C$, then f is a normal function.*

By "continuous" here we mean continuous in the extended sense, that is, we consider the image of f to be in the metric space consisting of the Riemann sphere with the usual chordal metric. Thus, for example, the function $f(z) = 1/(1 - z)$ is continuous in $D \cup C$.

LEMMA 4. *Let f be an analytic function in D and let $p \in C$ be such that there exist two sequences of points $\{z_n\}$ and $\{z'_n\}$ in D with the properties $z_n \to p$, $z'_n \to p$, $f(z_n) \to 0$, $f(z'_n) \to \infty$, and $|(z_n - z'_n)/(1 - \overline{z'_n} z_n)| < 1/2$. Then p is a nonnormal point for f.*

LEMMA 5. *Let f be an analytic function in D and suppose that there exist two positive constants α and β such that $\alpha < |f(z_2)/f(z_1)| < \beta$ whenever z_1 and z_2 are points of D satisfying $|(z_2 - z_1)/(1 - \overline{z_1} z_2)| < 1/2$. Then f is a normal function.*

Finally, we give a result about bounded analytic functions which will be useful to us.

LEMMA 6. *Let $B(z)$ be an analytic function with $|B(z)| < 1$ for $z \in D$. If z_1 and z_2 are points in D such that $|(z_2 - z_1)/(1 - \overline{z_1} z_2)| < 1/2$, then*

$$1/3 < |(1 - B(z_2))/(1 - B(z_1))| < 3.$$

PROOF. From Schwarz's Lemma, we have, for $z_1, z_2 \in D$,

$$|B(z_2) - B(z_1)| \le |1 - \overline{B(z_1)} B(z_2)|\, |(z_2 - z_1)/(1 - \overline{z_1} z_2)|.$$

The triangle inequality gives

$$|\,|1 - B(z_1)| - |1 - B(z_2)|\,| \le |B(z_1) - B(z_1)|$$

and

$$\begin{aligned} |1 - \overline{B(z_1)} B(z_2)| &= |(1 - \overline{B(z_1)}) + \overline{B(z_1)}(1 - B(z_2))| \\ &\le |1 - B(z_1)| + |1 - B(z_2)|. \end{aligned}$$

Thus, for $|(z_2 - z_1)/(1 - \overline{z_1}z_2)| < 1/2$, we have

$$||1 - B(z_1)| - |1 - B(z_2)|| < (1/2)(|1 - B(z_1)| + |1 - B(z_2)|).$$

Letting $\theta = |1 - B(z_2)|/|1 - B(z_1)|$, this last inequality becomes

$$|\theta - 1| < (1/2)(\theta + 1)$$

which implies $1/3 < \theta < 3$, which is the desired result.

3. We are now in a position to present the main results.

THEOREM 1. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $\{a_n\}$ is a bounded monotone sequence of real numbers, then $f(z)$ is a normal function.*

PROOF. Let $g(z) = (1 - z)f(z) = a_0 + \sum_{n=1}^{\infty} b_n z^n$, where $b_n = a_n - a_{n-1}$. If $b = \lim_{n\to\infty} a_n$, then $\sum_{n=1}^{\infty} |b_m| = |a_0 - b|$ so g is bounded and continuous on $D \cup C$, and $g(1) = b$. If $b \neq 0$, then $f(z) = g(z)/(1 - z)$ is continuous on $D \cup C$ (where $f(1) = \infty$), and f is a normal function by Lemma 3.

If $b = 0$, and $a_0 = 0$, then $f(z) \equiv 0$ and the result is trivial.

If $b = 0$ and $a_0 \neq 0$, then $g(z) = a_0(1 + \sum_{n=1}^{\infty}(b_n/a_0)z^n)$ and g is continuous on $D \cup C$. Let $B(z) = -\sum_{n=1}^{\infty}(b_n/a_0)z^n$. Then $|B(z)| < 1$ for $z \in D$ and $g(z) = a_0(1 - B(z))$. If $z_1, z_2 \in D$ are such that $|(z_2 - z_1)/(1 - \overline{z_1}z_2)| < 1/2$, then

$$1/3 < |g(z_2)/g(z_1)| < 3$$

by Lemma 6. Also, for $|(z_2 - z_1)/(1 - \overline{z_1}z_2)| < 1/2$, we have

$$|z_1 - z_2| < (1/2)|1 - \overline{z_1}z_2| < (1/2)(|1 - z_1| + |1 - z_2|)$$

and thus

$$|1 - z_2| \leq |1 - z_1| + |z_1 - z_2| < (3/2)|1 - z_1| + (1/2)|1 - z_2|$$

which means that

$$|1 - z_2| \leq 3|1 - z_1|.$$

Since the situation is symmetric in z_1 and z_2, we have

$$1/3 < |1 - z_1|/|1 - z_2| < 3.$$

Hence, for $|(z_2 - z_1)/(1 - \overline{z_1}z_2)| < 1/2$, we obtain

$$1/9 < |f(z_2)/f(z_1)| = |g(z_2)/g(z_1)|\,|(1 - z_1)/(1 - z_2)| < 9.$$

Thus, f is a normal function by Lemma 5.

COROLLARY. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where either $a_n\overline{a_{n+1}} > 0$ for each n or $a_n\overline{a_{n+1}} < 0$ for each n, and $\{|a_n|\}$ is a bounded monotone sequence, then f is a normal function.*

PROOF. By hypotheses, there exists a fixed $\theta \in [0, 2\pi)$ such that $b_n = e^{i\theta}a_n$ is a real number for each n. Let $f_1(z) = e^{i\theta}f(z) = \sum_{n=0}^{\infty} b_n z^n$. If $a_n\overline{a_{n+1}} = b_n b_{n+1} > 0$ for each n and $\{|b_n|\}$ is a bounded monotone sequence, then f_1 is a normal function by Theorem 1. If $a_n\overline{a_{n+1}} = b_n b_{n+1} < 0$ for each n, then $f_2(z) = f_1(-z)$ is a normal function by Theorem 1. In either case, that f is a normal function follows directly from the definition of a normal function.

THEOREM 2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *where* $\sum_{n=1}^{\infty} |a_n - a_{n-1}| < \infty$ *and* $\lim_{n\to\infty} a_n \neq 0$. *Then* f *is a normal function.*

PROOF. Let $g(z) = (1-z)f(z) = a_0 + \sum_{n=1}^{\infty} b_n z^n$, where, as before, $b_n = a_n - a_{n-1}$. Then g is continuous in $D \cup C$, and $g(1) = a_0 + \sum_{n=1}^{\infty} b_n = \lim_{n\to\infty} a_n \neq 0$. Thus $f(z) = g(z)/(1-z)$ is continuous on $D \cup C$ (where $f(1) = \infty$) and thus f is a normal function by Lemma 3.

REMARK 1. Under the hypotheses of Theorem 1 or Theorem 2, the proofs show that the function $f(z)$ is bounded in a neighborhood of each point of $C - \{1\}$. This fact will be used in the construction of examples.

REMARK 2. It is an open question whether Theorem 2 is true when $a_n \to 0$. This question, in turn, appears to depend upon the manner in which $g(z)$ can approach 0 as $z \to 1$, and little appears to be known about this.

We note that, in view of Lemma 2, the change of a finite number of coefficients of a function will not change whether or not the function is a normal function. Thus, Theorem 1 could be restated taking this into account. In addition, the addition of a bounded function to a normal function results in a normal function, and this allows for additional variations on the hypotheses of Theorem 1 (and possibly Theorem 2). We will not pursue such statements here. However, the proof of Theorem 2 can be modified to produce a somewhat stronger statement, as follows.

THEOREM 3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *and let there exist a positive integer* k *such that both* $\sum_{n=k}^{\infty} |a_n - a_{n-k}| < \infty$ *and*

$$\sum_{j=0}^{k-1} \left(\lim_{n\to\infty} a_{nk+j}\right) w^j \neq 0 \quad \text{for each } w \text{ with } w^k = 1.$$

Then $f(z)$ *is a normal function.*

PROOF. Let $g(z) = (1 - z^k)f(z) = \sum_{n=0}^{\infty} b_n z^n$, where $b_n = a_n$ for $0 \leq n \leq k-1$, and $b_n = a_n - a_{n-k}$ for $n \geq k$. Then, for $w^k = 1$,

$$g(w) = \sum_{n=0}^{\infty} b_n w^n = \sum_{j=0}^{k-1} (\lim a_{nk+j}) w^j \neq 0$$

and g is continuous in $D \cup C$ since $\sum |b_n| < \infty$. It follows that $f(z)$ is continuous in $D \cup C$ (with $f(w) = \infty$ for $w^k = 1$) so f is a normal function by Lemma 3.

For example, if $a_n = 1 + (-1)^n (1 - 1/n)$, then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a normal function by Theorem 3 (with $k = 2$).

4. We now give some examples to show the delicate nature of the results of the previous section. We note that Mathews has given an example of a nonnormal function whose coefficients converge to zero [**5**, Example 1, p. 28]. (Actually, Mathews claimed only that his example has bounded coefficients, but it is easily verified that his example is an H^2 function.) By a similar construction, we give an example of a nonnormal function with nonnegative coefficients such that the coefficients converge to zero. This shows that the monotonicity required in Theorem 1 cannot be easily discarded.

EXAMPLE 1. *There exists a nonnormal function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *such that* $a_n \geq 0$ *for each* n *and* $\lim_{n\to\infty} a_n = 0$.

PROOF. Let $z'_n = (1 - e^{-2n})i$ and $z_n = (1 - \frac{1}{2}e^{-2n})i$, $n = 1, 2, 3, \ldots$, and let $B_1(z)$ be the Blaschke product with zeros at $\{z'_n\}$. If $B(z) = B_1(z)\overline{B_1(\overline{z})}$, then $B(z)$ is real valued on the real axis, $B(z) = 0$ for $z = z'_n$ and for $z = \overline{z'_n}$ for each n, and there exists a $\delta > 0$ such that $|B(z)| > \delta$ for $z = z_n$ and $z = \overline{z_n}$ for each n (see [2, pp. 11–13]). Let $h(z) = \log(1/(1 + z^2))$ and let $g(z) = h(z)B(z)$. Then $g(z)$ is real valued on the real axis, so $g(z) = \sum_{n=0}^{\infty} b_n z^n$, where each b_n is a real number. Further $|g(z)| \leq |h(z)|$ and h is an H^2 function, so $\sum_{n=0}^{\infty} |b_n|^2 < \infty$ and thus $\lim_{n\to\infty} b_n = 0$. Finally, $g(z_n) \to \infty$, while $g(z'_n) = 0$ for each n. It follows that i is a nonnormal point for g by Lemma 4, for it is easily verified that $|(z_n - z'_n)/(1 - \overline{z'_n} z_n)| \to 1/3$ as $n \to \infty$. (It is also easily verified that the only other nonnormal point for g is $-i$, for g is continuous on $(D \cup C) - \{i, -i\}$, although we will not use this fact in what follows.)

Now let $c_n = \sup\{|b_k|: k \geq n\}$. Then $\{c_n\}$ is a decreasing sequence and $c_n \to 0$. If $G(z) = \sum_{n=0}^{\infty} c_n z^n$, then, by Remark 1 following Theorem 2, G is bounded in a neighborhood of i and so i is a nonnormal point for $g + G$. But then the function $f(z) = g(z) + G(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n = b_n + c_n \geq 0$, has a nonnormal point at i, and so f is not a normal function by Lemma 1. Thus f is the desired function.

Campbell and Piranian [3] have given an example at a bounded function $f(z) = \sum a_n z^n$ such that the function $g(z) = \sum |a_n| z^n$ is not a normal function. Mathews [5, Example 2, p. 30] gave an example of a Bloch function $f(z) = \sum a_n z^n$ for which $g(z) = \sum |a_n| z^n$ is not a Bloch function, but it is not clear whether $g(z)$ is a normal function or not. Here, we look in the opposite direction and give an example of a nonnormal function $f(z) = \sum a_n z^n$ for which $g(z) = \sum |a_n| z^n$ is a function satisfying the hypotheses of Theorem 1.

EXAMPLE 2. *There exists a nonnormal analytic function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *such that* $|a_n| = 1$ *for each* n.

PROOF. Let $g(z)$ be the function constructed in the proof of Example 1. Then $g(z) = \sum_{n=0}^{\infty} b_n z^n$, where b_n is a real number for each n and i is a nonnormal point for g. We may assume that $|b_n| \leq 1$, for we can simply omit any coefficient with modulus greater than 1 without changing any of the listed properties of g. For each n, let $d_n = \sqrt{1 - b_n^2}$ and note that

$$d_n - d_{n-1} = (b_{n-1}^2 - b_n^2) \Big/ \left(\sqrt{1 - b_n^2} + \sqrt{1 - b_{n-1}^2} \right)$$

so that, since $b_n \to 0$,

$$|d_n - d_{n-1}| \leq b_n^2 + b_{n-1}^2$$

for n sufficiently large. Since $\sum b_n^2 < \infty$, we have that $\sum |d_n - d_{n-1}| < \infty$ and $d_n \to 1$ since $b_n \to 0$. Thus, $F(z) = \sum_{n=0}^{\infty} i d_n z^n$ is a normal function by Theorem 2 and F is bounded in a neighborhood of i by Remark 1 following Theorem 2. Thus $f(z) = g(z) + F(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies $a_n = b_n + i d_n$ and so $|a_n| = 1$ for each n. Further i is a nonnormal point for f by Lemma 2, so f is not a normal function.

REMARK 3. Although coefficients in Example 2 have constant modulus, it is easy to modify these coefficients so that their moduli are strictly increasing (or decreasing). For, if $\{e_n\}$ is a sequence such that $\sum |e_n| < \infty$, $\arg e_n = \arg a_n$ and $\{|e_n|\}$ is strictly decreasing. Then the subtraction (or addition) of the function

$H(z) = \sum_{n=0}^{\infty} e_n z^n$ from (to) the function $f(z)$ in Example 2 will give the desired result.

It would be of interest to be able to construct an example of a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\{|a_n|\}$ converges monotonically to zero but $f(z)$ is not a normal function. Thusfar, we have been unable to construct such an example. Such an example would be especially interesting if each a_n were a real number.

5. In this section we deal briefly with the situation in which the coefficients are monotone and unbounded.

We first note that Theorem 1 is no longer valid if the condition of boundedness of the coefficients is removed.

EXAMPLE 3. *There exists a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\{a_n\}$ is an increasing sequence of real numbers and f is not a normal function.*

PROOF. Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be a nonnormal function with nonnegative coefficients such that $b_n < 1$ for each n, and i is a nonnormal point for g. Such a function was constructed in Example 1. Setting $f(z) = g(z) + (1/(1-z))^2 = \sum_{n=0}^{\infty}(n+1+b_n)z^n$, we have that i is a nonnormal point for f (since $1/(1-z)^2$ is bounded near i) and $\{a_n = n+1+b_n\}$ is an increasing sequence. Thus f is the desired function.

However, despite this example, it is possible to state conditions on an increasing sequence of coefficients under which the function is a normal function.

THEOREM 4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in D, where $\{a_n\}$ is an increasing sequence of real numbers. If there exists a positive integer k such that the sequence $\{b_n\}$ is a bounded monotone sequence, where $b_0 = a_0$, $b_1 = a_1 - ka_0, \ldots, b_n = \sum_{j=0}^{k^*(n)}(-1)^j\binom{k}{j}a_{n-j}$, where $k^*(n) = \min\{k, n\}$, then f is a normal function.*

PROOF. Let $f_k(z) = (1-z)^k f(z) = \sum_{n=0}^{\infty} b_n z^n$. From the proof of Theorem 1, f_k has the form

$$f_k(z) = g_k(z)/(1-z),$$

where g_k is continuous on $D \cup C$. If $g_k(1) \neq 0$, then $f(z) = g_k(z)/(1-z)^{k+1}$ is continuous on $D \cup C$ (with $f(1) = \infty$) and so f is normal by Lemma 3. If $g_k(1) = 0$, then by the method of proof of Theorem 1 we have if $z_1, z_2 \in 0$ and $|z_2 - z_1|/|1 - \overline{z_1}z_2| < 1/2$ then

$$1/3 < |g_k(z_2)/g_k(z_1)| < 3$$

and

$$1/3 < |1-z_2|/|1-z_1| < 3,$$

which yields

$$1/3^{k+2} < |f(z_2)/f(z_1)| < 3^{k+2},$$

so that f is a normal function by Lemma 5.

For example, if

$$f(z) = \sum_{n=1}^{\infty}\left(n - 1 + \left(\frac{1}{\sqrt{n+1}}\right)\right)z^n,$$

then for $k = 1$ we have $b_0 = 0$, $b_1 = 1/\sqrt{2}$, and $b_n = 1 + (1/\sqrt{n+1}) - (1/\sqrt{n})$ for $n \geq 2$, and $\{b_n\}$ is an increasing sequence which converges to 1. Thus, f is a normal function.

References

1. J. M. Anderson, J. Clunie and Ch. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **270** (1974), 12–37.
2. F. Bagemihl and W. Seidel, *Sequential and continuous limits of meromorphic functions*, Ann. Acad. Sci. Fenn. Ser. AI **280** (1960), 1–17.
3. D. M. Campbell and G. Piranian, *Problems on the normality of holomorphic functions*, Houston J. Math. (to appear).
4. O. Lehto and K. I. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math. **97** (1957), 47–65.
5. J. H. Mathews, *Coefficients of uniformly normal-Bloch functions*, Yokohama Math. J. **21** (1973), 27–31.
6. J. J. Neitzke, *Coefficients of Bloch functions*, Ph.D. Dissertation, Michigan State Univ., 1980.
7. L. R. Sons and D. M. Campbell, *Hadamard gap series and normal functions*, Bull. London Math. Soc. **12** (1980), 115–118.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824