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A SOBOLEV INEQUALITY FOR PLURIHARMONIC FUNCTIONS

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ABSTRACT. A Sobolev inequality is proved which implies that, on a smooth bounded domain D contained in $\mathbf{C}^n$, the L^2 inner product of two pluriharmonic functions is defined whenever one of them is in $C^\infty(\bar{D})$ and the other is dominated by some negative power of the distance to the boundary.

Suppose D is a smooth bounded domain in $\mathbf{C}^n$. We shall prove that for each positive integer s, there is a constant $C = C(s)$ such that

$$(1.1) \qquad \left|\int_D fg\right| \leq C\|f\|_s\|g\|_{-s}$$

for all pluriharmonic functions f and g in $L^2(D)$. Here $\|f\|_s^2 = \sum_{|\alpha|\leq s}\int_D |D^\alpha f|^2$ is the usual Sobolev s-norm and

$$\|g\|_{-s} = \operatorname{Sup}\left\{\left|\int_D g\phi\right| : \phi \in C_0^\infty(D); \|\phi\|_s = 1\right\}.$$

This Sobolev inequality implies that for each positive integer s, there is an integer $m = m(s)$ and a constant $K = K(s)$ such that for pluriharmonic functions f and g in $L^2(D)$,

$$(1.2) \qquad \left|\int_D fg\right| \leq K\left(\operatorname*{Sup}_{\substack{|\alpha|\leq m\\ z\in D}} |D^\alpha f(z)|\right)\left(\operatorname*{Sup}_{z\in D} |g(z)|\, d(z)^s\right)$$

where $d(z)$ is equal to the distance of z to bD, the boundary of D. Inequality (1.2) is a simple consequence of (1.1) because $\|g\|_{-s-n-1} \leq c\operatorname{Sup}\{|g(z)|\, d(z)^s : z \in D\}$ for pluriharmonic functions g (see [**2**]).

Inequality (1.1) is known to hold for holomorphic functions f and g (see [**1**]) and has proved to be very useful in the study of boundary behavior of holomorphic mappings. It is the purpose of this note to extend (1.1) to pluriharmonic functions.

Let $W^s(D)$ denote the usual Sobolev space of complex valued functions on D with derivatives up to order s in $L^2(D)$. Let $W_0^s(D)$ be the closure of $C_0^\infty(D)$ in $W^s(D)$ and let $P^s(D)$ be the subspace of $W^s(D)$ consisting of pluriharmonic functions. A function u is said to vanish to order t on bD if $D^\alpha u(z) = 0$ for all multi-indices α with $|\alpha| \leq t$ and all $z \in bD$. It is easy to verify that a function in $C^t(\bar{D})$ which vanishes to order $t-1$ on bD is in $W_0^t(D)$.

We shall prove (1.1) by constructing a linear operator L^s which is bounded from $P^s(D)$ to $W_0^s(D)$ such that $\int_D fg = \int_D (L^s f)g$ for all pluriharmonic functions g in $L^2(D)$. Then (1.1) follows because when f and g are pluriharmonic and in $L^2(D)$,

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we obtain

$$\left|\int_D fg\right| = \left|\int_D (L^s f)g\right| \le \|L^s f\|_s \|g\|_{-s} \le C\|f\|_s\|g\|_{-s}.$$

Construction of the operators L^s. Let r be a C^∞ function such that $D = \{r < 0\}, bD = \{r = 0\}$ and $dr \neq 0$ on bD. For $u \in C^\infty(\bar{D})$, we set

$$L^s u = u - \Delta\left(\sum_{k=0}^{s-1} \theta_k r^{k+2}\right) \tag{1.3}$$

where the functions θ_k are to be determined inductively. Notice that the function acted upon by the Laplacian in (1.3) vanishes to second order on bD, and therefore that $\int_D ug = \int_D (L^s u)g$ when g is pluriharmonic by integration by parts.

Let X be a C^∞ function which is equal to 1 in a neighborhood of bD and zero in a neighborhood of $\{|\nabla r| = 0\}$. Define $L^1 u = u - \Delta(\theta_0 r^2)$ where $\theta_0 = \frac{1}{2}X|\nabla r|^{-2}u$. The function θ_0 has been chosen in this way so that $L^1 u$ vanishes on bD, i.e., so that $L^1 u \in W_0^1(D)$.

Suppose $\theta_0, \theta_1, \ldots, \theta_{t-1}$ have been chosen so that $L^t u$ vanishes to order $t-1$ on bD. We set $L^{t+1}u = L^t u - \Delta(\theta_t r^{t+2})$ where θ_t is to be determined so that $L^{t+1}u$ vanishes to order t on bD. Note that no matter what we choose θ_t to be, $L^{t+1}u$ vanishes to order $t-1$ on bD. Let $(\partial/\partial n)$ be a vector field which points in the normal direction on bD. For convenience, let us take $(\partial/\partial n) = (\nabla r \cdot \nabla)/|\nabla r|^2$. We set

$$\theta_t = \frac{X}{(t+2)!}|\nabla r|^{-2}\left(\frac{\partial}{\partial n}\right)^t L^t u$$

so that $(\partial/\partial n)^t L^{t+1}u = 0$ on bD. This guarantees that $L^{t+1}u$ vanishes to order t on bD and hence that $L^{t+1}u \in W_0^{t+1}(D)$. The induction is complete.

To finish the proof, we must show that L^s is bounded from $P^s(D)$ to $W_0^s(D)$. We shall use the shorthand notation:

$$\partial^\alpha \bar{\partial}^\beta = \left(\frac{\partial^\alpha}{\partial z^\alpha}\right)\left(\frac{\partial^\beta}{\partial \bar{z}^\beta}\right).$$

A simple induction reveals that L^s is a linear differential operator of the form

$$\sum_{|\alpha|+|\beta| \le k \le N_s} A_{\alpha,\beta,k} r^k \partial^\alpha \bar{\partial}^\beta \tag{1.4}$$

where $N_s = \frac{s}{2}(s+3)$ and $A_{\alpha,\beta,k} \in C^\infty(\bar{D})$. We now prove that any operator of the form (1.4) maps $P^s(D)$ to $W^s(D)$ boundedly. If f is a pluriharmonic function in $C^\infty(\bar{D})$, $|\alpha| + |\beta| \le k$, $A \in C^\infty(\bar{D})$, and D^γ is a derivative of order $|\gamma| \le s$, then

$$\int_D |D^\gamma(A r^k \partial^\alpha \bar{\partial}^\beta f)|^2 = \sum_{|\omega|+|\delta|+m=2|\gamma|} \int_D A_{\omega,\delta} r^{2k-m} D^\omega \partial^\alpha \bar{\partial}^\beta f D^\delta \bar{\partial}^\alpha \partial^\beta \bar{f}$$
$$= \sum_{\substack{|\omega| \le |\gamma| \\ |\delta| \le |\gamma|}} \int_D \tilde{A}_{\omega,\delta} D^\omega f D^\delta \bar{f} \le C\|f\|_s^2.$$

Here we have used integration by parts and the fact that $\partial^\rho \bar{\partial}^\rho f = 0$ for any multi-index ρ. The power of r in the integrand is exactly large enough so that derivatives from $\partial^\alpha\bar\partial^\beta$ can be taken off f via integration by parts until there are $|\gamma|$ or less derivatives remaining.

We have now obtained that there is a constant C such that $\|L^s f\|_s \leq C\|f\|_s$ for pluriharmonic functions f in $C^\infty(\bar D)$. It remains for us to prove that this inequality also holds for a general function f in $P^s(D)$. To do this, we shall approximate f in $W^s(D)$ by functions of the form $f_\epsilon(z) = \sum_{j=0}^m \phi_j f(z - \epsilon\nu_j)$ where ϵ is a small number greater than zero, $\{\nu_j\}_{j=1}^m$ is a set of unit vectors in $\mathbf{C}^n$, and $\{\phi_j\}_{j=0}^m$ is a partition of unity of $\bar D$ such that $\phi_0 \in C_0^\infty(D)$ and $\nu_0 = 0$, and such that for $j \geq 1$, $\operatorname{Supp}\phi_j$ intersects bD in a small enough set so that the unit vector ν_j points out of D and intersects bD transversally at all points in a neighborhood of $\operatorname{Supp}\phi_j \cap bD$. The partition $\{\phi_j\}$ and the vectors $\{\nu_j\}$ can be chosen independently of f. Note that $L^s f_\epsilon$ can be written in the form

$$\sum_{j=1}^{m} \sum_{|\alpha|+|\beta|\leq k\leq N_s} A^j_{\alpha,\beta,k} r^k \partial^\alpha\bar\partial^\beta f(z-\epsilon\nu_j)$$

where the functions $A^j_{\alpha,\beta,k}$ have support contained in $\operatorname{Supp}\phi_j$, are in $C^\infty(\bar D)$, and do not depend on f. Now the Sobolev s-norm of each term

$$A^j_{\alpha,\beta,k} r^k \partial^\alpha\bar\partial^\beta f(\cdot-\epsilon\nu_j)$$

can be estimated exactly as before since $f(\cdot - \epsilon\nu_j)$ is pluriharmonic in a neighborhood of $\operatorname{Supp}\phi_j \cap \bar D$. This procedure yields the estimate,

$$\|L^s f_\epsilon\|_s \leq C\sum \|f(\cdot-\epsilon\nu_j)\|_{W^s(\operatorname{Supp}\phi_j\cap D)} \leq mC\|f\|_s.$$

Hence, for small $\epsilon > 0$, the functions $L^s f_\epsilon$ form a bounded set in $W^s(D)$. The Banach-Saks theorem [**3**, p. 181] implies that there is a sequence of numbers $\epsilon_i > 0$ with $\epsilon_i \to 0$ as $i \to \infty$ such that the arithmetic means of the sequence $\{L^s f_{\epsilon_i}\}_{i=1}^\infty$ converge in $W^s(D)$ to some function v in $W^s(D)$. It is clear that $v = L^s f$, and therefore that $\|L^s f\|_s \leq mC\|f\|_s$. This completes the proof.

REMARK. The inequality (1.1) suggests the definition of a special norm on pluriharmonic functions given by

$$|||f|||_s = \operatorname{Sup}\left\{\left|\int_D fg\right| : \|g\|_{-s} = 1; g \in P(D)\right\},$$

where $P(D)$ denotes the space of L^2 pluriharmonic functions. It is conceivable that the norm $|||f|||_s$ is equivalent to the norm $\|f\|_s$ for f in $P^s(D)$. However, a proof of this fact requires detailed knowledge of the regularity properties of the operator which projects $L^2(D)$ onto its subspace $P(D)$ consisting of pluriharmonic functions.

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