

SOME REMARKS ON q -BETA INTEGRAL

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ABSTRACT. The following q -integral

$$\int_{-c}^d \frac{(-qt/c)_{\alpha-1}(qt/d)_{\beta-1}}{(-qet)_{\alpha+\beta}} d_q t$$

is evaluated. A more general q -integral is also considered. Some applications to the q -Wilson (or Askey-Wilson) polynomials are also given.

1. Introduction. In a recent paper [1] Andrews and Askey gave the q -beta integral

$$\begin{aligned} (1.1) \quad & \int_{-c}^d \frac{(-qx/c; q)_{\infty}(qx/d; q)_{\infty}}{(-q^{\alpha}x/c; q)_{\infty}(q^{\beta}x/d; q)_{\infty}} d_q x \\ &= \frac{cd(1-q)(q; q)_{\infty}(q^{\alpha+\beta}; q)_{\infty}(-c/d; q)_{\infty}(-d/c; q)_{\infty}}{(c+d)(q^{\alpha}; q)_{\infty}(q^{\beta}; q)_{\infty}(-q^{\beta}c/d; q)_{\infty}(-q^{\alpha}d/c; q)_{\infty}} \\ &= \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{cd}{c+d} \frac{(-c/d; q)_{\infty}(-d/c; q)_{\infty}}{(-q^{\beta}c/d; q)_{\infty}(-q^{\alpha}d/c; q)_{\infty}}. \end{aligned}$$

The last equality holds for $0 < q < 1$. For notation we refer the reader to [1]. However, since base q is not changed throughout this work we shall write $(a)_n$ for $(a; q)_n$. If, in addition, we write $(z)_{\alpha} = (z)_{\infty}/(zq^{\alpha})_{\infty}$ then (1.1) can be written more compactly (and perhaps more suggestively) as

$$(1.2) \quad \int_{-c}^d (-qx/c)_{\alpha-1}(qx/d)_{\beta-1} d_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{cd}{c+d} \left(-\frac{c}{d}\right)_{\beta} \left(-\frac{d}{c}\right)_{\alpha}.$$

In this note we first generalize (1.2) to obtain the q -beta formula

$$(1.3) \quad \int_{-c}^d \frac{(-qt/c)_{\alpha-1}(qt/d)_{\beta-1}}{(-qet)_{\alpha+\beta}} d_q t = \frac{cd}{c+d} \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{(-d/c)_{\alpha}(-c/d)_{\beta}}{(-qed)_{\alpha}(-qec)_{\beta}}$$

which is a q -analog of a formula of Dinghas [4, Formula (2.4)] and which reduces to the Andrews-Askey formula (1.1) when $e = 0$. We shall obtain this formula first by using the transformation formula of Sears [4, (5.2)] and then by another method.

We next consider the still more general integral

$$(1.4) \quad I = \int_{-c}^d \frac{(-qt/c)_{\alpha-1}(qt/d)_{\beta-1}}{(-qet/c)_{\gamma}(qft/d)_{\delta}} d_q t$$

for $\alpha + \beta = \gamma + \delta$. We show that I can be written as a well-poised ${}_8\Phi_7$ (see formula (3.1) below).

Finally we use (1.4) to give q -integral representation for the general q -Wilson [2] ${}_4\Phi_3$ polynomials.

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2. Formula (1.3). Sears proved the identity [5, (5.2)]

$$(2.1) \quad \frac{(e)_\infty(f)_\infty}{(a)_\infty(b)_\infty(c)_\infty} {}_3\Phi_2 \left[\begin{matrix} a, b, c; q \\ e, f \end{matrix} \right] - \frac{q}{e} \frac{(q^2/e)_\infty(qf/e)_\infty}{(qa/e)_\infty(qb/e)_\infty(qc/e)_\infty} \\ \cdot {}_3\Phi_2 \left[\begin{matrix} qa/e, qb/e, qc/e; q \\ q^2/e, qf/e \end{matrix} \right] = \frac{(e)_\infty(q/e)_\infty(f/a)_\infty(f/b)_\infty(f/c)_\infty}{(a)_\infty(b)_\infty(c)_\infty(qa/e)_\infty(qb/e)_\infty(qc/e)_\infty}$$

where $abcq = ef$.

If in this formula we put $a = -dq^\alpha/c$, $b = q^\beta$, $c = -edq$, $e = -dq/c$ and $f = -edq^{\alpha+\beta+1}$ we get, after some manipulation, formula (1.3).

3. The integral I . Let $\alpha + \beta = \gamma + \delta$ and consider the q -integral I in (1.4). By definition we can write

$$I = \frac{(-dq/c)_{\alpha-1}(q)_{\beta-1}d(1-q)}{(fq)_\delta(-edq/c)_\gamma} {}_4\Phi_3 \left[\begin{matrix} -q^\alpha d/c, q^\beta, -edq/c, fq; q \\ fq^{1+\delta}, -dq/c, -edq^{1+\gamma}/c \end{matrix} \right] \\ + c(1-q) \frac{(q)_{\alpha-1}(-cq/d)_{\beta-1}}{(eq)_\gamma(-cfq/d)_\delta} {}_4\Phi_3 \left[\begin{matrix} q^\alpha, -cfq/d, -cq^\beta/d, eq; q \\ eq^{1+\gamma}, -cq/d, -cfq^{1+\delta}/d \end{matrix} \right].$$

Now making use of the q -analog of Whipple's theorem [3, p. 69], which expresses a well-poised ${}_8\Phi_7$ as a combination of two Saalschützian ${}_4\Phi_3$, we get

$$(3.1) \quad I = (c+d)(1-q)(q)_\infty(-cq/d)_{\beta-1}(-dq/c)_{\alpha-1}/(eq)_{\gamma-\alpha} \\ \cdot {}_8\Phi_7 \left[\begin{matrix} A, q\sqrt{A}, -q\sqrt{A}, q^\beta, fq, -edq/c, -fcq^{\delta-\alpha+1}/d, eq^{\gamma-\alpha+1}, q^\alpha \\ \sqrt{A}, -\sqrt{A}, efq^2, eq^{1+\beta}, -fcq^{1+\beta}/d, -edq^{\alpha+\beta-\delta+1}/c, fq^{\alpha-\gamma+1} \end{matrix} \right]$$

where $A = efq^{1+\beta}$.

Formula (2.1) also implies (1.3) by taking $f = \delta = 0$ and using the formula

$${}_2\Phi_1[a, b; c; c/ab] = \frac{(c/a)_\infty(c/b)_\infty}{(c)_\infty(a/ab)_\infty}.$$

Returning to (3.1), we now recall a transformation of Bailey [3, p. 70]

$${}_8\Phi_7 \left[\begin{matrix} A, q\sqrt{A}, -q\sqrt{A}, B, C, D, E, F; A^2q^2/BDEF \\ \sqrt{A}, -\sqrt{A}, qA/B, qA/C, qA/D, qA/E, qA/F \end{matrix} \right] \\ = \frac{(Aq)_\infty(Aq/DE)_\infty(Aq/DF)_\infty(Aq/EF)_\infty}{(Aq/D)_\infty(Aq/E)_\infty(Aq/F)_\infty(Aq/DEF)_\infty} {}_4\Phi_3 \left[\begin{matrix} Aq/BC, D, E, F; q \\ DEF/A, Aq/B, Aq/C \end{matrix} \right]$$

where the series on the RHS terminates whereas the series on the L.H.S. may not terminate.

With the aid of this transformation we see that

$$(3.2) \quad I = (c+d)(1-q) \frac{(q)_\infty(-cq/d)_{\beta-1}(-dq/c)_{\alpha-1}}{(eq)_{\gamma-\alpha}} \\ \cdot \frac{(-fcq/d)_\beta(-edq^{1+\alpha-\delta}/c)_\beta}{(efq^2)_\beta(q^{\alpha-\delta})_\beta} \cdot {}_4\Phi_3 \left[\begin{matrix} q^\delta, q^\beta, -edq/c, -fcq^{\delta-\alpha+1}/d; q \\ q^{1+\delta-\alpha}, eq^{1+\beta}, fq^{1+\delta} \end{matrix} \right]$$

provided that either q^δ , q^β , $-edq/c$, or $-fcq^{\delta-\alpha+1}/d$ is of the form q^{-n} where $n = 0, 1, 2, 3, \dots$

This formula suggests the following representation for the q -Wilson polynomial

$$W_n(x) = {}_4\Phi_3 \left[\begin{matrix} q^{-n}, abq^{1+n}, q^{-x}, cdq^{1+x}; q \\ aq, cq, bdq \end{matrix} \right];$$

namely,

$$(3.3) \quad W_n(x) = \frac{(aq^{1+x})_\infty (cdq^{1+x})_\infty (q^{-x})_\infty (bq^{-x}/c)_\infty (aq^{-x}/cd)_\infty (bdq^{1+x})_\infty}{(1-q)(q)_\infty (aq)_\infty (dq^{1+x} - q^{-x}/c)(q^{-n}/c)_n (q^{-2x}/cd)_\infty (cdq^{2+2x})_\infty} \cdot \frac{1}{(bdq)_\infty (abq^{1+n}/c)_\infty} \int_{c^{-1}q^{-x}}^{dq^{1+x}} \frac{(abtq^{1+n})_\infty (tq^{-n})_\infty (ctq^{1+x})_\infty (tq^{-x}/d)_\infty}{(bt)_\infty (at/d)_\infty (ct)_\infty (t)_\infty} d_q t$$

or the representation,

$$(3.4) \quad W_n(x) = \frac{(q^{-n}/b)_\infty (d^2q^{2+x})_\infty (1/c)_\infty (aq^{1+x})_\infty (abq^{1+n})_\infty (bq^{-x}/c)_\infty (q^{-x})_\infty}{(1-q)(q)_\infty (aq)_\infty (bq^{n-x})_\infty (bdq)_\infty (q^{x+1-n}/b)_\infty} \cdot \frac{q^{-x}}{(q^{-x}/c)_\infty (abq^{1+n}/c)_\infty} \cdot \int_{bq^n}^{q^x} \frac{(tq^{1-x})_\infty (tq^{1-n}/b)_\infty (atq^{1-x}/c)_\infty (dtq^{1-n})_\infty}{(tdq)_\infty (tq^{-x-n}/b)_\infty (tqa)_\infty (tq^{-x-n}/c)_\infty} d_q t.$$

The latter implies the representation for the q -Meixner polynomial

$$(3.5) \quad {}_2\Phi_1 \left[\begin{matrix} q^{-n}, q^{-x}; bq^{1+n} \\ cq \end{matrix} \right] = \frac{q^{-x}(q^{-x})_\infty (bq^{-x}/c)_\infty (1/c)_\infty (q^{-n}/b)_\infty}{(1-q)(q)_\infty (q^{-x}/c)_\infty (bq^{n-x})_\infty (q^{1+x-n}/b)_\infty} \cdot \int_{bq^n}^{q^x} \frac{(tq^{1-x})_\infty (tq^{1-n}/b)_\infty}{(tq^{-x-n}/b)_\infty (tq^{-x-n}/c)_\infty} d_q t.$$

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