SOME REMARKS ON \(q\)-BETA INTEGRAL

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ABSTRACT. The following \(q\)-integral
\[
\int_{-c}^{d} \frac{(-qt/c)_{\alpha-1}(qt/d)_{\beta-1}}{(-qet)_{\alpha+\beta}} \, dq \, t
\]
is evaluated. A more general \(q\)-integral is also considered. Some applications to the \(q\)-Wilson (or Askey-Wilson) polynomials are also given.

1. Introduction. In a recent paper [1] Andrews and Askey gave the \(q\)-beta integral
\[
\int_{-c}^{d} \frac{(-qx/c; q)_{\infty}(qx/d; q)_{\infty}}{(-q^2x/c; q)_{\infty}(q^2x/d; q)_{\infty}} \, dq \, x
\]
(1.1)
\[
= \frac{cd(1-q)(q; q)_{\infty}(q^a+b; q)_{\infty}(-c/d; q)_{\infty}(-d/c; q)_{\infty}}{(c+d)(q^a; q)_{\infty}(q^b; q)_{\infty}(-q^0c/d; q)_{\infty}(-q^0d/c; q)_{\infty}} \\
= \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{cd}{c+d} \frac{(-c/d; q)_{\infty}(-d/c; q)_{\infty}}{(-q^0c/d; q)_{\infty}(-q^0d/c; q)_{\infty}}
\]
The last equality holds for \(0 < q < 1\). For notation we refer the reader to [1]. However, since base \(q\) is not changed throughout this work we shall write \((a)_n\) for \((a; q)_n\). If, in addition, we write \((z)_\alpha = (z)_{\infty}/(z^{qa})_{\infty}\) then (1.1) can be written more compactly (and perhaps more suggestively) as

\[
(1.2) \quad \int_{-c}^{d} (-qx/c)_\alpha (-qx/d)_\beta \, dq \, t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{cd}{c+d} \frac{(-c/d)_\beta}{(-d/c)_\alpha}
\]

In this note we first generalize (1.2) to obtain the \(q\)-beta formula

\[
(1.3) \quad \int_{-c}^{d} \frac{(-qt/c)_{\alpha-1}(qt/d)_{\beta-1}}{(-qet)_{\alpha+\beta}} \, dq \, t = \frac{cd}{c+d} \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{(-d/c)_\alpha(-c/d)_\beta}{(-qed)_\beta(-qec)_\alpha}
\]
which is a \(q\)-analog of a formula of Dinghas [4, Formula (2.4)] and which reduces to the Andrews-Askey formula (1.1) when \(e = 0\). We shall obtain this formula first by using the transformation formula of Sears [4, (5.2)] and then by another method.

We next consider the still more general integral

\[
(1.4) \quad I = \int_{-c}^{d} \frac{(-qt/c)_{\alpha-1}(qt/d)_{\beta-1}}{(-qet/c)_{\gamma}} \, dq \, t
\]
for \(\alpha + \beta = \gamma + \delta\). We show that \(I\) can be written as a well-poised \(8\Phi_7\) (see formula (3.1) below).

Finally we use (1.4) to give \(q\)-integral representation for the general \(q\)-Wilson [2] \(4\Phi_3\) polynomials.
2. **Formula (1.3).** Sears proved the identity \[5, (5.2)\]

\[
\frac{(e)_{\infty} (f)_{\infty}}{(a)_{\infty} (b)_{\infty} (c)_{\infty}} \cdot \begin{array}{c}
\Phi_2 \\
\end{array}
\left[
\begin{array}{c}
a, b, c; q \\
e, f
\end{array}
\right] - \frac{q}{e} \frac{(q^2/e)_{\infty} (q/e)_{\infty}}{(q/a/e)_{\infty} (q/b/e)_{\infty} (q/c/e)_{\infty}}
\]

(2.1)

\[
\cdot \begin{array}{c}
\Phi_2 \\
\end{array}
\left[
\begin{array}{c}
qa/e, qb/e, qc/e; q \\
q^2/e, qf/e
\end{array}
\right] = \frac{(e)_{\infty} (f/e)_{\infty}}{(a)_{\infty} (b)_{\infty} (c)_{\infty}} \cdot \begin{array}{c}
\Phi_2 \\
\end{array}
\left[
\begin{array}{c}
qa/e, qb/e, qc/e; q \\
q^2/e, qf/e
\end{array}
\right]
\]

where \(abcq = ef\).

If in this formula we put \(a = -dq^\alpha/c, b = q^\beta, c = -dq, e = -dq/c\) and \(f = -edq^\alpha+\beta+1\) we get, after some manipulation, formula (1.3).

3. **The integral \(I.** Let \(\alpha + \beta = \gamma + \delta\) and consider the \(q\)-integral \(I\) in (1.4). By definition we can write

\[
I = \frac{(-dq/c)_{\infty}(q^\alpha q^\beta)(1-q)}{(fq)_{\infty} (-deq/c)_{\infty} q^{(\gamma - \delta)(\alpha + \beta + 1)}}
\]

\[
\cdot \begin{array}{c}
\Phi_3 \\
\end{array}
\left[
\begin{array}{c}
-q^\alpha d/c, q^\beta, -edq/c, fq; q \\
fq^{1+\delta}, -dq/c, -deq^{1+\gamma}/c
\end{array}
\right] + c(1-q)_{\infty} \frac{(cq/d)_{\infty}(q^\alpha q^\beta)(1-q)}{\gamma, -cq/d, -cq^\beta/d, eq; q}
\]

\[
\cdot \begin{array}{c}
\Phi_3 \\
\end{array}
\left[
\begin{array}{c}
q^\alpha, -cfq/d, -cq^\beta/d, eq; q \\
eq^{1+\gamma} - cq/d, -cfq^{1+\delta}/d
\end{array}
\right]
\]

Now making use of the \(q\)-analog of Whipple’s theorem \([3, p. 69]\), which expresses a well-poised \(\Phi_7\) as a combination of two Saalschützian \(\Phi_4\), we get

\[
I = (c + d)(1-q)(q)_{\infty}(cq/d)_{\infty}(-dq/c)_{\infty}(eq)^{-\alpha - \gamma}
\]

(3.1)

\[
\cdot \begin{array}{c}
\Phi_7 \\
\end{array}
\left[
\begin{array}{c}
A, q\sqrt{A}, -q\sqrt{A}, q^\beta, f, q, -edq/c, -fcdq^{-\alpha+1}/d, eq^{-\alpha+1}; q^\alpha \\
\sqrt{A}, -\sqrt{A}, efq^2, eq^{1+\beta}, -fcq^{1+\beta}/d, -edq^\alpha+\beta^{-\delta+1}/c, f^{q^{1+\delta}}
\end{array}
\right]
\]

where \(A = efq^{1+\beta}\).

Formula (2.1) also implies (1.3) by taking \(f = \delta = 0\) and using the formula

\[
2\Phi_1[a, b; c; c/ab] = \frac{(c/a)_{\infty}(c/b)_{\infty}}{(c)_{\infty}(a/c)_{\infty}}
\]

Returning to (3.1), we now recall a transformation of Bailey \([3, p. 70]\)

\[
\cdot \begin{array}{c}
\Phi_7 \\
\end{array}
\left[
\begin{array}{c}
A, q\sqrt{A}, -q\sqrt{A}, B, C, D, E, F; A^2q^2/BDEF \\
\sqrt{A}, -\sqrt{A}, qA/B, qA/C, qA/D, qA/E, qA/F
\end{array}
\right]
\]

\[
= \frac{(Aq)_{\infty}(DE)_{\infty}(DF)_{\infty}(EF)_{\infty}}{(Aq)_{\infty}(DE)_{\infty}(DF)_{\infty}(EF)_{\infty}} \cdot \begin{array}{c}
\Phi_3 \\
\end{array}
\left[
\begin{array}{c}
Aq/BC, D, E, F; q \\
DE/A, Aq/B, Aq/C
\end{array}
\right]
\]

where the series on the RHS terminates whereas the series on the L.H.S. may not terminate.

With the aid of this transformation we see that

\[
I = (c + d)(1-q)(q)_{\infty}(cq/d)_{\infty}(-dq/c)_{\infty}(eq)^{-\alpha - \gamma}
\]

(3.2)

\[
\cdot \begin{array}{c}
\Phi_3 \\
\end{array}
\left[
\begin{array}{c}
q^\delta, q^\beta, -edq/c, -fcq^{\alpha+1}/d; q \\
q^{1+\delta-\alpha}, eq^{1+\beta}, f^{q^{1+\delta}}
\end{array}
\right]
\]

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provided that either \( q^6 \), \( q^9 \), \(-edq/c\), or \(-fcq^{\mu-x-1}/d\) is of the form \( q^{-n} \) where \( n = 0, 1, 2, 3, \ldots \).

This formula suggests the following representation for the \( q \)-Wilson polynomial

\[
W_n(x) = 4 \Phi_3 \left[ q^{-n}, abq^{1+n}, q^{-x}, cdq^{1+x}; q \right],
\]

namely,

\[
W_n(x) = \frac{(aq^{1+z})_{\infty} (cdq^{1+z})_{\infty} (q^{-x})_{\infty} (bq^{-x}/c)_{\infty} (aq^{-x}/cd)_{\infty} (bdq^{1+z})_{\infty}}{(1-q)(q)(aq)_{\infty} (dq^{1+z} - q^{-x}/c) (q^{-n}/c)_{n} (q^{-2x}/cd)_{\infty} (cdq^{2+2z})_{\infty}}
\]

\[
\cdot \frac{1}{(bdq)_{\infty} (aq^{1+n}/c)_{\infty}} \int_{c^{-1}q^{-z}}^{d q^{1+z}} \frac{(abtq^{1+n})_{\infty} (tq^{-n})_{\infty} (ctq^{1+z})_{\infty} (tq^{-x}/d)_{\infty}}{(bt)_{\infty} (at/d)_{\infty} (ct)_{\infty}} dq t
\]

or the representation,

\[
W_n(x) = \frac{(q^{-n}/b)_{\infty} (d^2q^{2+x})_{\infty} (1/c)_{\infty} (aq^{1+z})_{\infty} (abq^{1+n})_{\infty} (bq^{-x}/c)_{\infty} (q^{-x})_{\infty}}{(1-q)(q)(aq)_{\infty} (bq^{n-x})_{\infty} (bdq)_{\infty} (q^{x+1-n}/b)_{\infty}} q^{-x}
\]

\[
\cdot \frac{q^{-x}/c)_{\infty} (aq^{1+n}/c)_{\infty}}{aq} \int_{bq^n}^{q^z} \frac{(tq^{1-x})_{\infty} (tq^{1-n}/b)_{\infty} (atq^{1-x}/c)_{\infty} (dtq^{1-n})_{\infty}}{(tq)_{\infty} (tq^{-x-n}/b)_{\infty} (tq^{-x-n}/c)_{\infty}} dq t.
\]

The latter implies the representation for the \( q \)-Meixner polynomial

\[
\Phi_2 \left[ q^{-n}, q^{-x}; bq^{1+n}; cq \right] = \frac{q^{-x} (q^{-n}/b)_{\infty} (bq^{-x}/c)_{\infty} (1/c)_{\infty} (q^{-n}/b)_{\infty}}{(1-q)(q)(aq)_{\infty} (bq^{n-x})_{\infty} (q^{1+x-n}/b)_{\infty}} q^{-x}
\]

\[
\cdot \int_{bq^n}^{q^z} \frac{(tq^{1-x})_{\infty} (tq^{1-n}/b)_{\infty}}{(tq^{-x-n}/b)_{\infty} (tq^{-x-n}/c)_{\infty}} dq t.
\]

REFERENCES


2. R. Askey and J. Wilson, A set of orthogonal polynomials that generalize the Racah coefficients of \( 6-j \) symbols, Mathematics Research Center, University of Wisconsin-Madison, MRC Technical Summary Report #1833.


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