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SOME REMARKS ON q-BETA INTEGRAL

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ABSTRACT. The following q-integral

$$\int_{-c}^{d} \frac{(-qt/c)_{\alpha-1}(qt/d)_{\beta-1}}{(-qet)_{\alpha+\beta}}\, d_q t$$

is evaluated. A more general q-integral is also considered. Some applications to the q-Wilson (or Askey-Wilson) polynomials are also given.

1. Introduction. In a recent paper [**1**] Andrews and Askey gave the q-beta integral

$$\begin{aligned}
&\int_{-c}^{d} \frac{(-qx/c;q)_\infty (qx/d;q)_\infty}{(-q^\alpha x/c;q)_\infty (q^\beta x/d;q)_\infty}\, d_q x \\
(1.1)\qquad &= \frac{cd(1-q)(q;q)_\infty (q^{\alpha+\beta};q)_\infty (-c/d;q)_\infty (-d/c;q)_\infty}{(c+d)(q^\alpha;q)_\infty (q^\beta;q)_\infty (-q^\beta c/d;q)_\infty (-q^\alpha d/c;q)_\infty} \\
&= \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{cd}{c+d} \frac{(-c/d;q)_\infty (-d/c;q)_\infty}{(-q^\beta c/d;q)_\infty (-q^\alpha d/c;q)_\infty}.
\end{aligned}$$

The last equality holds for $0 < q < 1$. For notation we refer the reader to [**1**]. However, since base q is not changed throughout this work we shall write $(a)_n$ for $(a;q)_n$. If, in addition, we write $(z)_\alpha = (z)_\infty/(zq^\alpha)_\infty$ then (1.1) can be written more compactly (and perhaps more suggestively) as

$$(1.2)\qquad \int_{-c}^{d} (-qx/c)_{\alpha-1}(qx/d)_{\beta-1}\, d_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{cd}{c+d} \left(-\frac{c}{d}\right)_\beta \left(-\frac{d}{c}\right)_\alpha.$$

In this note we first generalize (1.2) to obtain the q-beta formula

$$(1.3)\qquad \int_{-c}^{d} \frac{(-qt/c)_{\alpha-1}(qt/d)_{\beta-1}}{(-qet)_{\alpha+\beta}}\, d_q t = \frac{cd}{c+d} \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{(-d/c)_\alpha (-c/d)_\beta}{(-qed)_\alpha (-qec)_\beta}$$

which is a q-analog of a formula of Dinghas [**4**, Formula (2.4)] and which reduces to the Andrews-Askey formula (1.1) when $e = 0$. We shall obtain this formula first by using the transformation formula of Sears [**4**, (5.2)] and then by another method.

We next consider the still more general integral

$$(1.4)\qquad I = \int_{-c}^{d} \frac{(-qt/c)_{\alpha-1}(qt/d)_{\beta-1}}{(-qet/c)_\gamma (qft/d)_\delta}\, d_q t$$

for $\alpha+\beta = \gamma+\delta$. We show that I can be written as a well-poised ${}_8\Phi_7$ (see formula (3.1) below).

Finally we use (1.4) to give q-integral representation for the general q-Wilson [**2**] ${}_4\Phi_3$ polynomials.

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2. Formula (1.3). Sears proved the identity [**5**, (5.2)]

$$\frac{(e)_\infty(f)_\infty}{(a)_\infty(b)_\infty(c)_\infty}\,{}_3\Phi_2\begin{bmatrix} a,b,c;q \\ e,f \end{bmatrix} - \frac{q}{e}\,\frac{(q^2/e)_\infty(qf/e)_\infty}{(qa/e)_\infty(qb/e)_\infty(qc/e)_\infty} \tag{2.1}$$
$$\cdot\,{}_3\Phi_2\begin{bmatrix} qa/e,qb/e,qc/e;q \\ q^2/e,qf/e \end{bmatrix} = \frac{(e)_\infty(q/e)_\infty(f/a)_\infty(f/b)_\infty(f/c)_\infty}{(a)_\infty(b)_\infty(c)_\infty(qa/e)_\infty(qb/e)_\infty(qc/e)_\infty}$$

where $abcq = ef$.

If in this formula we put $a = -dq^\alpha/c$, $b = q^\beta$, $c = -edq$, $e = -dq/c$ and $f = -edq^{\alpha+\beta+1}$ we get, after some manipulation, formula (1.3).

3. The integral I. Let $\alpha+\beta = \gamma+\delta$ and consider the q-integral I in (1.4). By definition we can write

$$I = \frac{(-dq/c)_{\alpha-1}(q)_{\beta-1}d(1-q)}{(fq)_\delta(-deq/c)_\gamma}\,{}_4\Phi_3\begin{bmatrix} -q^\alpha d/c, q^\beta, -edq/c, fq; q \\ fq^{1+\delta}, -dq/c, -deq^{1+\gamma}/c \end{bmatrix}$$
$$+ c(1-q)\frac{(q)_{\alpha-1}(-cq/d)_{\beta-1}}{(eq)_\gamma(-cfq/d)_\delta}\,{}_4\Phi_3\begin{bmatrix} q^\alpha, -cfq/d, -cq^\beta/d, eq; q \\ eq^{1+\gamma}, -cq/d, -cfq^{1+\delta}/d \end{bmatrix}.$$

Now making use of the q-analog of Whipple's theorem [**3**, p. 69], which expresses a well-poised ${}_8\Phi_7$ as a combination of two Saalschützian ${}_4\Phi_3$, we get

(3.1)

$$I = (c+d)(1-q)(q)_\infty(-cq/d)_{\beta-1}(-dq/c)_{\alpha-1}/(eq)_{\gamma-\alpha}$$
$$\cdot\,{}_8\Phi_7\begin{bmatrix} A, q\sqrt{A}, -q\sqrt{A}, q^\beta, fq, -edq/c, -fcq^{\delta-\alpha+1}/d, eq^{\gamma-\alpha+1}; q^\alpha \\ \sqrt{A}, -\sqrt{A}, efq^2, eq^{1+\beta}, -fcq^{1+\beta}/d, -edq^{\alpha+\beta-\delta+1}/c, fq^{\alpha-\gamma+1} \end{bmatrix}$$

where $A = efq^{1+\beta}$.

Formula (2.1) also implies (1.3) by taking $f = \delta = 0$ and using the formula

$${}_2\Phi_1[a,b;c;c/ab] = \frac{(c/a)_\infty(c/b)_\infty}{(c)_\infty(a/ab)_\infty}.$$

Returning to (3.1), we now recall a transformation of Bailey [**3**, p. 70]

$${}_8\Phi_7\begin{bmatrix} A, q\sqrt{A}, -q\sqrt{A}, B, C, D, E, F; A^2q^2/BDEF \\ \sqrt{A}, -\sqrt{A}, qA/B, qA/C, qA/D, qA/E, qA/F \end{bmatrix}$$
$$= \frac{(Aq)_\infty(Aq/DE)_\infty(Aq/DF)_\infty(Aq/EF)_\infty}{(Aq/D)_\infty(Aq/E)_\infty(Aq/F)_\infty(Aq/DEF)_\infty}\,{}_4\Phi_3\begin{bmatrix} Aq/BC, D, E, F; q \\ DEF/A, Aq/B, Aq/C \end{bmatrix}$$

where the series on the RHS terminates whereas the series on the L.H.S. may not terminate.

With the aid of this transformation we see that

$$I = (c+d)(1-q)\frac{(q)_\infty(-cq/d)_{\beta-1}(-dq/c)_{\alpha-1}}{(eq)_{\gamma-\alpha}} \tag{3.2}$$
$$\cdot\,\frac{(-fcq/d)_\beta(-edq^{1+\alpha-\delta}/c)_\beta}{(efq^2)_\beta(q^{\alpha-\delta})_\beta}\cdot{}_4\Phi_3\begin{bmatrix} q^\delta, q^\beta, -edq/c, -fcq^{\delta-\alpha+1}/d; q \\ q^{1+\delta-\alpha}, eq^{1+\beta}, fq^{1+\delta} \end{bmatrix}$$

provided that either q^δ, q^β, $-edq/c$, or $-fcq^{\delta-\alpha+1}/d$ is of the form q^{-n} where $n = 0, 1, 2, 3, \ldots$.

This formula suggests the following representation for the q-Wilson polynomial

$$W_n(x) = {}_4\Phi_3\left[\begin{matrix} q^{-n}, abq^{1+n}, q^{-x}, cdq^{1+x}; q \\ aq, cq, bdq \end{matrix}\right];$$

namely,

(3.3)

$$W_n(x) = \frac{(aq^{1+x})_\infty (cdq^{1+x})_\infty (q^{-x})_\infty (bq^{-x}/c)_\infty (aq^{-x}/cd)_\infty (bdq^{1+x})_\infty}{(1-q)(q)_\infty (aq)_\infty (dq^{1+x} - q^{-x}/c)(q^{-n}/c)_n (q^{-2x}/cd)_\infty (cdq^{2+2x})_\infty}$$
$$\cdot \frac{1}{(bdq)_\infty (abq^{1+n}/c)_\infty} \int_{c^{-1}q^{-x}}^{dq^{1+x}} \frac{(abtq^{1+n})_\infty (tq^{-n})_\infty (ctq^{1+x})_\infty (tq^{-x}/d)_\infty}{(bt)_\infty (at/d)_\infty (ct)_\infty (t)_\infty} \, d_q t$$

or the representation,

(3.4)

$$W_n(x) = \frac{(q^{-n}/b)_\infty (d^2q^{2+x})_\infty (1/c)_\infty (aq^{1+x})_\infty (abq^{1+n})_\infty (bq^{-x}/c)_\infty (q^{-x})_\infty}{(1-q)(q)_\infty (aq)_\infty (bq^{n-x})_\infty (bdq)_\infty (q^{x+1-n}/b)_\infty}$$
$$\cdot \frac{q^{-x}}{(q^{-x}/c)_\infty (abq^{1+n}/c)_\infty}$$
$$\cdot \int_{bq^n}^{q^x} \frac{(tq^{1-x})_\infty (tq^{1-n}/b)_\infty (atq^{1-x}/c)_\infty (dtq^{1-n})_\infty}{(tdq)_\infty (tq^{-x-n}/b)_\infty (tqa)_\infty (tq^{-x-n}/c)_\infty} \, d_q t.$$

The latter implies the representation for the q-Meixner polynomial

$$(3.5)\quad {}_2\Phi_1\left[\begin{matrix} q^{-n}, q^{-x}; bq^{1+n} \\ cq \end{matrix}\right] = \frac{q^{-x}(q^{-x})_\infty (bq^{-x}/c)_\infty (1/c)_\infty (q^{-n}/b)_\infty}{(1-q)(q)_\infty (q^{-x}/c)_\infty (bq^{n-x})_\infty (q^{1+x-n}/b)_\infty}$$
$$\cdot \int_{bq^n}^{q^x} \frac{(tq^{1-x})_\infty (tq^{1-n}/b)_\infty}{(tq^{-x-n}/b)_\infty (tq^{-x-n}/c)_\infty} \, d_q t.$$

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