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ON DUNFORD-PETTIS OPERATORS THAT ARE PETTIS-REPRESENTABLE

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ABSTRACT. Let E be a Banach space. It is shown that if every Dunford-Pettis operator $T: L^1[0,1] \to E^*$ is Pettis-representable, then every operator $T: L^1[0,1] \to E^*$ is Pettis-representable.

It is well known that a bounded linear operator T from $L^1[0,1]$ into a Banach space E that is Bochner or Pettis-representable is a Dunford-Pettis operator, i.e., T maps weakly convergent sequences into norm convergent sequences. It was suspected for a while that any Dunford-Pettis operator $T: L^1[0,1] \to L^1[0,1]$ is Bochner representable, but Costé (see [**3**, p. 90]) gave an example of a convolution type operator $T: L^1[0,1] \to L^1[0,1]$ that is a Dunford-Pettis operator but is not Bochner representable. Costé's example suggested the following problem: If every Dunford-Pettis operator from $L^1[0,1]$ into a Banach space E is Bochner representable, is every bounded linear operator $T: L^1[0,1] \to E$ Bochner representable? In [**2**] Bourgain solved the above problem affirmatively. A parallel problem to the one solved by Bourgain can now be asked as follows: If every Dunford-Pettis operator T from $L^1[0,1]$ to a Banach space E is Pettis-representable, is every bounded linear operator $T: L^1[0,1] \to E$ Pettis-representable?

In [**10**] we showed that the answer to the above problem is positive when the Banach space E is complemented in a Banach lattice. In fact under this hypothesis one can conclude that every bounded linear operator $T: L^1[0,1] \to E$ is Bochner representable (see [**5**]).

In this paper, we will show that the answer is also positive for dual Banach spaces, indeed we shall show that if every Dunford-Pettis operator T from L^1 into a dual Banach space E^* is Pettis-representable then every bounded linear operator is Pettis-representable.

By an operator between two Banach spaces, we always mean a bounded linear operator. All the notions used in this paper and not defined can be found in [**3, 5**]. Let E be a Banach space and let T be an operator from $L^1[0,1]$ to E. The operator T is said to be Bochner (resp. Pettis) representable if there exists $g: [0,1] \to E$ Bochner-integrable and essentially bounded (resp. Pettis-integrable and scalarly essentially bounded) such that for every f in $L^1[0,1]$, $T(f) = \text{Bochner-}\int_0^1 f \cdot g \, d\lambda$ (resp., $\text{Pettis-}\int_0^1 f \cdot g \, d\lambda$).

A Banach space E is said to have the Radon-Nikodym property (RNP) (resp., the weak-Radon-Nikodym property (WRNP)), if every operator $T: L^1[0,1] \to E$ is Bochner-representable (resp., Pettis-representable). For more about the RNP and the WRNP see [**3, 5, 7, 8, 9**].

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THEOREM 1. *Let E be a Banach space. If every Dunford-Pettis operator $T\colon L^1[0,1] \to E^*$ is Pettis-representable then E does not contain any isomorphic copy of l_1.*

PROOF. If not, then l_1 is isomorphic to a subspace of E. Let $S\colon l^1 \to E$ be the embedding of l_1 into E and let V be a quotient map from l_1 onto $C[0,1]$ the space of continuous functions on the interval $[0,1]$ and let $V^* = U$ be the adjoint map $U\colon M[0,1] \to l^\infty$ where $M[0,1] = C[0,1]^*$ the Banach space of all Radon measures on $[0,1]$.

It is clear that U is an embedding. The space $M[0,1]$ is an L-space, therefore, by a result of Grothendieck [**6**] the operator U has a lifting Q from $M[0,1]$ to E^* such that $S^*Q = U$. It is evident that Q is an embedding. Let R be the natural embedding of $L^1[0,1]$ into $M[0,1]$, and let T be a Dunford-Pettis operator from $L^1[0,1]$ to $L^1[0,1]$ that is not Bochner-representable in $L^1[0,1]$ [**3**, p. 92]. Consider the following diagram

$$\begin{tikzcd} L^1[0,1] \arrow[d, "T"'] & {[0,1]} \arrow[r, "g_1"] \arrow[d, "g_2"] & E^* \arrow[d, "S^*"] \\ L^1[0,1] \arrow[r, "R"] & M[0,1] \arrow[r, "U"] \arrow[ur, "Q"] & l_\infty \end{tikzcd}$$

where g_1 and g_2 are to be specified shortly. The operator $QRT\colon L^1[0,1] \to E^*$ is Dunford-Pettis, therefore QRT is Pettis-representable. Let $g_1\colon [0,1] \to E^*$ be its Pettis-derivative, that is

$$QRT(f) = \text{Pettis-}\int_0^1 f g_1\, d\lambda \quad \text{for every } f \in L^1[0,1]. \tag{1}$$

Let g_2 be a weak*-derivative of the operator $RT\colon L^1[0,1] \to M[0,1]$, that is,

$$RT(f) = \omega^*\text{-}\int_0^1 f \cdot g_2\, d\lambda \quad \text{for every } f \in L^1[0,1] \tag{2}$$

and this means that $\langle \phi, RT(f)\rangle = \int_0^1 \langle \phi, g_2(t)\rangle d\lambda$ for every ϕ in $C[0,1]$ [**3**, VI.8.6]. The function g_2 takes its values in the weak* closure of the image by RT of the unit ball of $L^1[0,1]$. Notice that (1) implies that $S^*QRT(f) = \text{Pettis-}\int_0^1 fS^*g_1 d\lambda$ for every $f \in L^1[0,1]$. But $S^*Q = U$, hence

$$URT(f) = \text{Pettis-}\int_0^1 fS^*g_1\, d\lambda \quad \text{for every } f \in L^1[0,1]. \tag{3}$$

On the other hand (2) implies

$$URT(f) = \omega^*\text{-}\int_0^1 fUg_2\, d\lambda \quad \text{for every } f \in L_1[0,1] \tag{4}$$

because U is weak* to weak* continuous. (3) and (4) imply that for every $a \in l_1$ and every Lebesgue measurable subset A in $[0,1]$ $\int_A \langle a, S^*g_1\rangle\, d\lambda = \int_A \langle a, Ug_2\rangle\, d\lambda$. This implies that $S^*g_1 = Ug_2(t)$ λ-almost everywhere, because l_1 is separable. The function $S^*g_1\colon [0,1] \to l_\infty$ is Pettis-integrable, therefore, the function $Ug_2\colon [0,1] \to l_\infty$ is Pettis-integrable. Hence $g_2\colon [0,1] \to M[0,1]$ is Pettis-integrable because U is an embedding, hence

$$RT(f) = \text{Pettis-}\int_0^1 f \cdot g_2\, d\lambda \quad \text{for every } f \in L^1[0,1]. \tag{5}$$

Let P be the usual projection from $M[0,1]$ into $R(L^1[0,1])$ and

$$W = R^{-1} \colon R(L^1[0,1]) \to L^1[0,1].$$

The map $h = WPg_2 \colon [0,1] \to L_1[0,1]$ is weakly measurable, therefore it is strongly measurable by the Pettis measurability theorem [**3**, p. 42], hence it is Bochner-integrable because it is bounded. Now (5) implies that

$$T(f) = WPRT(f) = \text{Pettis-}\int_0^1 fWPg_2\, d\lambda$$
$$= \text{Bochner-}\int_0^1 f \cdot h\, d\lambda$$

for every $f \in L^1[0,1]$. This shows that T is Bochner-representable, a contradiction that finishes the proof.

Theorem 1 and the result of [**1**] or [**7**] give the following theorem.

THEOREM 2. *Let E be a Banach space. Then the following statements are equivalent:*

(i) *The space E does not contain a copy of l_1;*
(ii) *The space E^* has the weak Radon-Nikodym property;*
(iii) *Every operator $T\colon L^1[0,1] \to E^*$ is Pettis-representable;*
(iv) *Every Dunford-Pettis operator $T\colon L^1[0,1] \to E^*$ is Pettis-representable.*

PROOF. (i)→(ii) is due to Bourgain [**1**] and Janicka [**7**].

(ii)→(iii)→(iv) are evident.

(iv)→(i) is Theorem 1.

It is worth mentioning the well-known fact that (i), (ii) or (iii) of Theorem 2 is equivalent to the following statement: (v) Every operator $T\colon L^1[0,1] \to E^*$ is Dunford-Pettis [**9**].

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