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REFLEXIVITY OF OPERATOR SPACES

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ABSTRACT. For reflexive Banach spaces E and F (with E or F having the approximation property), the space of opeartors from E into F (the inductive tensor product of E^* with F) is reflexive if and only if the operator space coincides with the inductive tensor product of E^* with F. Consequently, E must be finite-dimensional if either the projective tensor product of E with E^* is reflexive, or the inductive tensor product of E with E^* is reflexive and E has the approximation property.

Introduction. Throughout, E and F will be reflexive Banach spaces, $E \otimes_\epsilon F$ the Banach space tensor product of E and F under the least cross norm ϵ (operator norm, norm giving the inductive topology), and $E \otimes_\pi F$ the Banach space tensor product of E and F under the greatest cross norm π (nuclear norm, norm giving the projective topology).

In this note we give solutions to two unsolved problems:

A. Does there exist an infinite-dimensional Banach space E such that $E \otimes_\epsilon E^*$ or $E \otimes_\pi E^*$ is reflexive?

B. Characterize those spaces $L(E, F)$ of operators from E into F which are reflexive.

Problem A was posed in [3] and is answered negatively for the π norm, and for the ϵ norm if E has the approximation property (a.p.). Grothendieck [2] characterized the reflexivity of $L(E, F)$ for $\mathfrak{S}$-topologies and made available the powerful results required for the norm topology when both E and F have the a.p. [2]. Thus, for that case, problem B was solved in [3] and [5]. Subsequently, Holub succeeded in removing the a.p. condition from one of the spaces and tied problem B to the theory of operators which attain their norm [4]. The characterization offered below is a.p.-independent.

Results.

THEOREM 1. *For reflexive Banach spaces E and F, $L(E, F)$ is reflexive if and only if $L(E, F) = E^* \otimes_\epsilon F$.*

PROOF. The sufficiency follows from [4, p. 176] since the proof therein of (ii) implies (i) does not assume that either space has the a.p.

If $L(E, F)$ is reflexive, then its closed subspace $E^* \otimes_\epsilon F$ is reflexive, and since ϵ is a reflexive cross norm, $(E^* \otimes_\epsilon F)^* = E \otimes_{\epsilon^0} F^*$ [6]. From [1, p. 72, 18] it follows that $E \otimes_\pi F^* | (E^* \otimes_\epsilon F)^0 = E \otimes_{\epsilon^0} F^*$, where the polar is taken in $E \otimes_\pi F^* = L(E, F)^*$. But

$$[E \otimes_\pi F^* | (E^* \otimes_\epsilon F)^0]^* = (E^* \otimes_\epsilon F)^{00},$$

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where the second polar is taken in $L(E,F)$. Thus

$$(E^* \otimes_\epsilon F)^{00} = (E \otimes_{\epsilon^0} F^*)^* = E^* \otimes_\epsilon F.$$

However,

$$L(E,F)|E^* \otimes_\epsilon F = (E^* \otimes_\epsilon F)^{00}$$

and the result follows.

COROLLARY 2. *$E \otimes_\pi F^*$ is reflexive if and only if $L(E,F) = E^* \otimes_\epsilon F$. Consequently, if $E \otimes_\pi E^*$ is reflexive, E is finite dimensional.*

PROOF. The first statement is a simple consequence of the theorem. If $L(E,E) = E^* \otimes_\epsilon E$, then the identity operator on E is compact and E must then be finite dimensional.

The following is accessible via several powerful theorems in [**2**]. The argument is actually shorter, however, by using a well-known result on polars and one of those theorems.

THEOREM 3. *If E and F are Banach spaces and one has the a.p., then $E^* \otimes_\epsilon F$ is reflexive if and only if $L(E,F) = E^* \otimes_\epsilon F$.*

PROOF. Since either E or F has the a.p., the canonical map $E \otimes_\pi F^* \to (E^* \otimes_\epsilon F)^*$ is an isometry into the second space [**2**, p. 179], and since $E^* \otimes_\epsilon F$ is reflexive, $(E^* \otimes_\epsilon F)^* = E \otimes_{\epsilon^0} F^*$ [**6**]. Thus,

$$(E \otimes_{\epsilon^0} F^*)^* |(E \otimes_\pi F^*)^0 = (E \otimes_\pi F^*)^* = L(E,F),$$

where the polar is taken in $(E \otimes_{\epsilon^0} F^*)^* = E^* \otimes_\epsilon F$. Since the polar is $\{0\}$, $E^* \otimes_\epsilon F = L(E,F)$.

There immediately follows

COROLLARY 4. *If E or F has the a.p., $E^* \otimes_\epsilon F$ is reflexive if and only if $E \otimes_\pi F^*$ is reflexive. Consequently, if $E^* \otimes_\epsilon E$ is reflexive then E is finite dimensional.*

Corollaries 2 and 3 thus solve problem III of [**3**] for the π norm, and the ϵ norm if E has the a.p.

THEOREM 5. *Let E and F be reflexive Banach spaces one of which has the a.p. The following are equivalent:*

(i) *$L(E,F)$ and $L(F,E)$ are reflexive;*

(ii) *every operator $T\colon E \to F$ and $S\colon F \to E$ is compact;*

(iii) *one of $E^* \otimes_\epsilon F$ or $E \otimes_\pi F^*$ is reflexive, and one of $E \otimes_\epsilon F^*$ or $E^* \otimes_\pi F$ is reflexive;*

(iv) *all of $E^* \otimes_\epsilon F$, $E \otimes_\pi F^*$, $E \otimes_\epsilon F^*$, and $E^* \otimes_\pi F$ are reflexive.*

PROOF. The equivalence of (i) and (ii) was shown in [**4**]. Theorem 3 and Corollary 4 give the equivalence of (i), (iii), and (iv).

REMARK. We will say that the pair of spaces E and F of Theorem 5 has P if they have any one of the four equivalent properties. The existence of a pair of infinite-dimensional spaces having property P is an unsolved problem. This problem was first posed in the setting of statement (ii) by Pelczynski at the Sopot conference for arbitrary E and F, and in the setting of (i) by Holub [**3**].

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