REFLEXIVITY OF OPERATOR SPACES

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ABSTRACT. For reflexive Banach spaces E and F (with E or F having the approximation property), the space of operators from E into F (the inductive tensor product of E* with F) is reflexive if and only if the operator space coincides with the inductive tensor product of E* with F. Consequently, E must be finite-dimensional if either the projective tensor product of E with E* is reflexive, or the inductive tensor product of E with E* is reflexive and E has the approximation property.

Introduction. Throughout, E and F will be reflexive Banach spaces, $E \otimes_\epsilon F$ the Banach space tensor product of E and F under the least cross norm $\epsilon$ (operator norm, norm giving the inductive topology), and $E \otimes_\pi F$ the Banach space tensor product of E and F under the greatest cross norm $\pi$ (nuclear norm, norm giving the projective topology).

In this note we give solutions to two unsolved problems:
A. Does there exist an infinite-dimensional Banach space E such that $E \otimes_\epsilon E^*$ or $E \otimes_\pi E^*$ is reflexive?
B. Characterize those spaces $L(E, F)$ of operators from E into F which are reflexive.

Problem A was posed in [3] and is answered negatively for the $\pi$ norm, and for the $\epsilon$ norm if E has the approximation property (a.p.). Grothendieck [2] characterized the reflexivity of $L(E, F)$ for $\sigma$-topologies and made available the powerful results required for the norm topology when both E and F have the a.p. [2]. Thus, for that case, problem B was solved in [3] and [5]. Subsequently, Holub succeeded in removing the a.p. condition from one of the spaces and tied problem B to the theory of operators which attain their norm [4]. The characterization offered below is a.p.-independent.

Results.

THEOREM 1. For reflexive Banach spaces E and F, $L(E, F)$ is reflexive if and only if $L(E, F) = E^* \otimes_\epsilon F$.

PROOF. The sufficiency follows from [4, p. 176] since the proof therein of (ii) implies (i) does not assume that either space has the a.p.

If $L(E, F)$ is reflexive, then its closed subspace $E^* \otimes_\epsilon F$ is reflexive, and since $\epsilon$ is a reflexive cross norm, $(E^* \otimes_\epsilon F)^* = E \otimes_{\epsilon^0} F^*$ [6]. From [1, p. 72, 18] it follows that $E \otimes_\pi F^*|(E^* \otimes_\epsilon F)^0 = E \otimes_{\epsilon^0} F^*$, where the polar is taken in $E \otimes_\pi F^* = L(E, F)^*$. But

$$[E \otimes_\pi F^*|(E^* \otimes_\epsilon F)^0]^* = (E^* \otimes_\epsilon F)^{00},$$

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where the second polar is taken in \( L(E, F) \). Thus
\[
(E^* \otimes_e F)^{00} = (E \otimes_{e0} F^*)^* =: E^* \otimes_e F.
\]
However,
\[
L(E, F)|E^* \otimes_e F = (E^* \otimes_e F)^{00}
\]
and the result follows.

**Corollary 2.** \( E \otimes_\pi F^* \) is reflexive if and only if \( L(E, F) = E^* \otimes_e F \). Consequently, if \( E \otimes_\pi E^* \) is reflexive, \( E \) is finite dimensional.

**Proof.** The first statement is a simple consequence of the theorem. If \( L(E, E) = E^* \otimes_e E \), then the identity operator on \( E \) is compact and \( E \) must then be finite dimensional.

The following is accessible via several powerful theorems in [2]. The argument is actually shorter, however, by using a well-known result on polars and one of those theorems.

**Theorem 3.** If \( E \) and \( F \) are Banach spaces and one has the a.p., then \( E^* \otimes_e F \) is reflexive if and only if \( L(E, F) = E^* \otimes_e F \).

**Proof.** Since either \( E \) or \( F \) has the a.p., the canonical map \( E \otimes_\pi F^* \rightarrow (E^* \otimes_e F)^* \) is an isometry into the second space [2, p. 179], and since \( E^* \otimes_e F \) is reflexive, \( (E^* \otimes_e F)^* = E \otimes_{e0} F^* \) [6]. Thus,
\[
(E \otimes_{e0} F^*)^* = (E \otimes_{e0} F^*)^0 = (E \otimes_{e0} F^*)^* = L(E, F),
\]
where the polar is taken in \( (E \otimes_{e0} F^*)^* = E^* \otimes_e F \). Since the polar is \( \{0\} \), \( E^* \otimes_e F = L(E, F) \).

There immediately follows

**Corollary 4.** If \( E \) or \( F \) has the a.p., \( E^* \otimes_e F \) is reflexive if and only if \( E \otimes_\pi F^* \) is reflexive. Consequently, if \( E^* \otimes_e E \) is reflexive then \( E \) is finite dimensional.

Corollaries 2 and 3 thus solve problem III of [3] for the \( \pi \) norm, and the \( \epsilon \) norm if \( E \) has the a.p.

**Theorem 5.** Let \( E \) and \( F \) be reflexive Banach spaces one of which has the a.p. The following are equivalent:

(i) \( L(E, F) \) and \( L(F, E) \) are reflexive;

(ii) every operator \( T: E \rightarrow F \) and \( S: F \rightarrow E \) is compact;

(iii) one of \( E^* \otimes_e F \) or \( E \otimes_\pi F^* \) is reflexive, and one of \( E \otimes_e F^* \) or \( E^* \otimes_\pi F \) is reflexive;

(iv) all of \( E^* \otimes_e F, E \otimes_\pi F^*, E \otimes_e F^* \), and \( E^* \otimes_\pi F \) are reflexive.

**Proof.** The equivalence of (i) and (ii) was shown in [4]. Theorem 3 and Corollary 4 give the equivalence of (i), (iii), and (iv).

**Remark.** We will say that the pair of spaces \( E \) and \( F \) of Theorem 5 has property \( P \) if they have any one of the four equivalent properties. The existence of a pair of infinite-dimensional spaces having property \( P \) is an unsolved problem. This problem was first posed in the setting of statement (ii) by Pelczynski at the Sopot conference for arbitrary \( E \) and \( F \), and in the setting of (i) by Holub [3].
REFERENCES


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