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CERTAIN APPLICATIONS OF THE THEORY OF POLAR-COMPOSITE POLYNOMIALS

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ABSTRACT. In a recent paper [5] the authors have, for the first time, given a detailed account of the theory of polar-composite polynomials in algebraically closed fields of characteristic zero. In another paper [6], we have given some applications of this theory and have obtained a few results for a new variety of composite polynomials which have been derived from certain polar-composite polynomials through iteration. In the present paper also we consider the same variety of composite polynomials, but our present study deals with a different aspect of application of the said theory. Besides other things, our main theorem here offers a generalization of a result due to Marden [2] (see also [1, Theorem (16,3)]).

1. Introduction. The notations and the terminology in the present text is borrowed from [6] (refer also to [5] for fuller details). We shall denote by $D(\mathbf{C}_\omega)$ the class of all generalized circular regions of $\mathbf{C}_\omega$, where $\mathbf{C}_\omega$ represents the projective field of the field $\mathbf{C}$ of complex numbers found by adjoining to $\mathbf{C}$, the scalar infinity ω. It is known (see [7, pp. 386–387, 352; 3, pp. 23, 35; or 4, p. 116]) that *the nontrivial generalized circular regions of $\mathbf{C}_\omega$ are the open interior (or exterior) of circles or the half-planes, with a connected subset (possibly empty) of their boundary adjoined.* We denote by $\mathcal{F}_n(\mathbf{C})$ the class of all nth-degree polynomials from $\mathbf{C}$ to $\mathbf{C}$ and by $Z(f)$ the set of all zeros of the polynomial f.

The following lemma, due to the authors [6, Lemma (2.1)], will play a significant role in the proof of our main theorem in the next section.

LEMMA (1.1). *Given $f \in \mathcal{F}_m(\mathbf{C})$, $\beta_1 \in \mathbf{C} - \{0\}$ and an integer p $(1 \leq p \leq m)$, let us define*

$$h_1(z) = \beta_1 f(z) + (-1)^p z^p f^{(p)}(z) \quad \textit{for all } z \in \mathbf{C}.$$

If $\beta_1 \neq \lambda^p$, where $\lambda = -(p!\,C(m,p))^{1/p} \exp(i\pi/p)$, and if $A \in D(\mathbf{C}_\omega)$ such that $Z(f) \subseteq A$, then every zero γ of $h_1(z)$ can be written in the form $\gamma = \alpha$ or $\gamma = \alpha[\beta_{1j}/(\beta_{1j} - \lambda)]$ for some $\alpha \in A$ and for some member β_{1j} for some j (the β_{1j}, $0 \leq j \leq p-1$, being the pth-roots of β_1).

2. Main theorem. In this section we consider the same class of composite polynomials $h(z)$ as was considered earlier (see [6, Theorem (2.2)]), but this time we use a different aspect of application of Lemma (1.1) in proving our main theorem of this paper. This theorem deals with the general problem of locating the zeros of the composite polynomials $h(z)$ referred above, but in a different manner, so as to include in it a result due to Marden (see [1, Theorem (16,3) or 2]). It may be

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pointed out that the proof of our main theorem here employs similar techniques as those of an earlier theorem due to the authors (see [**6**, Theorem (2.2)]) except that, while applying Lemma (1.1) at the proper stage in the proof, we utilize $\arg\gamma$ instead of $|\gamma|$.

THEOREM (2.1). *Given the polynomials*

$$f(z) = \sum_{k=0}^{m} a_k z^k, \qquad g(z) = \sum_{k=0}^{n} b_k z^k \qquad (a_m, b_n \neq 0)$$

from $\mathbf{C}$ *to* $\mathbf{C}$, *let us form the polynomial*

$$h(z) = \sum_{k=0}^{m} a_k g((-1)^{p-1} p!\, C(k,p)) z^k, \qquad 1 \le p \le m.$$

If all the zeros of $f(z)$ *lie in the sector*

$$S_0 \colon \mu_1 \le \arg z \le \mu_2, \qquad \mu_2 - \mu_1 = \mu < \pi,$$

and if the pth-roots of all the zeros of $g(z)$ *lie in the lune*

$$L \colon \theta_1 \le \arg[z/(z-\lambda)] \le \theta_2, \qquad |\theta_1| + |\theta_2| \le (\pi - \mu)/n,$$

where $\lambda = -(p!\, C(m,p))^{1/p} \exp(i\pi/p)$, *then all the zeros of* $h(z)$ *lie in the sector*

$$S_n \colon \mu_1 + \min(0, n\theta_1) \le \arg z \le \mu_2 + \max(0, n\theta_2).$$

PROOF. Let $\beta_1, \beta_2, \ldots, \beta_n$ be the zeros of $g(z)$. For each value of $k = 1, 2, \ldots, n$, if β_{kj} $(0 \le j \le p_1)$ denotes the pth-roots of β_k, the hypothesis on the zeros of $g(z)$ implies that

$$\theta_1 \le \arg[\beta_{kj} \backslash (\beta_{kj} - \lambda)] \le \theta_2, \qquad |\theta_1| + |\theta_2| \le (\pi - \mu)/n \tag{2.1}$$

for all values of k and j and, hence that $\beta_{kj} \neq 0, \lambda$ for any values of k or j. That is,

$$\beta_k \neq 0, \lambda^p \quad \forall k = 1, 2, \ldots, n. \tag{2.2}$$

If $\{g_k(z)\}$ and $\{h_k(z)\}$ are sequences of polynomials defined by

$$g_k(z) = b_n(\beta_1 - z)(\beta_2 - z)(\beta_2 - z)\cdots(\beta_k - z),$$
$$h_0(z) = f(z), \qquad h_k(z) = \beta_k h_{k-1}(z) + (-1)^p z^p h_{k-1}^{(p)}(z)$$

for $k = 1, 2, \ldots, n$, then we know (see [**6**, proof of Theorem (2.2)]) that

$$h_k \mathcal{F}_m(\mathbf{C}) \quad \forall k = 1, 2, \ldots, n \tag{2.3}$$

and that

$$h_n(z) = (-1)^n b_n^{-1} h(z), \qquad g_n(z) = (-1)^n g(z),$$

whatever originally be the value of n. To prove the theorem, it remains only to show that

$$Z(h_n) \subseteq S_n. \tag{2.4}$$

To this effect we consider the sector S_k $(k = 1, 2, \ldots, n)$ given by

$$S_k \colon \mu_1 + \min(0, k\theta_1) \le \arg z \le \mu_2 + \max(0, k\theta_2).$$

If we write

$$A_1 = \{z\colon \mu_1 \le \arg z \le \mu_1 + \pi\}, \qquad A_2 = \{z\colon \mu_2 - \pi \le \arg z \le \mu_2\},$$

then A_1, $A_2 \in D(\mathbf{C}_\omega)$ such that $Z(f) \subseteq S_0 = A_1 \cap A_2$, because $\mu_2 - \mu_1 = u < \pi$. In view of this fact and the relation (2.2), we see that Lemma (1.1) is applicable to the polynomial $h_1(z)$ in each of the cases when the set A_1 or A_2 is substituted for the set A in the lemma. Therefore, for each value of $i = 1, 2$, every zero γ of $h_1(z)$ can be written in the form $\gamma = \alpha$ or $\gamma = \alpha[\beta_{1j}/(\beta_{1j} - \lambda)]$ for some $\alpha \in A_i$ and for some β_{1j} for some j. Now, the definitions of the sets A_i and the inequalities (2.1) help us to conclude that $\arg\gamma$ must satisfy one of the inequalities

$$\mu_1 \le \arg\gamma \le \mu_1 + \pi \quad \text{or} \quad \mu_1 + \theta_1 \le \arg\gamma \le \mu_1 + \pi + \theta_2$$

as well as one of the inequalities

$$\mu_2 - \pi \le \arg\gamma \le \mu_2 \quad \text{or} \quad \mu_2 - \pi + \theta_1 \le \arg\gamma \le \mu_2 + \theta_2.$$

From this it follows that $\arg\gamma$ satisfies the inequalities

$$\mu_1 + \min(0, \theta_1) \le \arg\gamma \le \mu_1 + \pi + \max(0, \theta_2)$$

and

$$\mu_2 - \pi + \min(0, \theta_1) \le \arg\gamma \le \mu_2 + \max(0, \theta_2).$$

Since $\mu_2 - \mu_1 < \pi$ and since

$$\mu_1 + \min(0, \theta_1) \le \mu_1 \le \mu_2 \le \mu_2 + \max(0, \theta_2),$$

we have that

$$\mu_1 + \min(0, \theta_1) \le \arg\gamma \le \mu_2 + \max(0, \theta_2).$$

That is, (2.4) holds true for $n = 1$. Now assume that (2.4) holds for $n = k - 1$. Then $h_{k-1} \in \mathcal{F}_m(\mathbf{C})$, due to (2.3), and all the zeros of the polynomial h_{k-1} lie in the sector

$$S_0'\colon \mu_1' \le \arg z \le \mu_2',$$

where

$$\mu_1' = \mu_1 + \min(0, (k-1)\theta_1), \qquad \mu_2' = \mu_2 + \max(0, (k-1)\theta_2).$$

Since (2.2) holds, the definition of h_k and the statement (2.4) for $n = 1$ (the case already proved), with h_1 replaced by h_{k-1} and β_1 by β_k, imply that $Z(h_k) \subseteq S_1'$, where

$$S_1'\colon \mu_1' + \min(0, \theta_1) \le \arg z \le \mu_2' + \max(0, \theta_2).$$

Finally the relations

$$\mu_1' + \min(0, \theta_1) = \mu_1 + \min(0, k\theta_1)$$

and

$$\mu_2' + \max(0, \theta_2) = \mu_2 + \max(0, k\theta_2)$$

imply that $Z(h_k) \subseteq S_k$ and (2.4) holds true for $n = k$. Therefore, (2.4) has been established for all n by mathematical induction. This completes the proof.

The following corollary is a result due to Marden [**1**, Theorem (16,3)] (see also [**2**, Theorem II]).

COROLLARY (2.2). *With the notation of Theorem* (2.1), *if* $Z(f) \subseteq S_0$ *and* $Z(g) \subseteq L_0$, *where*

$$L_0 = \{z \in \mathbf{C}: \theta_1 \leq \arg[z/(z-m)] \leq \theta_2\}, \qquad |\theta_1| + |\theta_2| \leq (\pi - \mu)/n,$$

then all the zeros of the polynomial

$$H(z) = \sum_{k=0}^{m} a_k g(k) z^k$$

lie in the sector S_n.

PROOF. In Theorem (2.1) if we take $p = 1$, then we notice that $\lambda = m$, $L = L_0$ and that $h(z) = H(z)$. Our corollary is now immediate from Theorem (2.1).

Next, we give a new result as application of the main theorem.

THEOREM (2.3). *If all the zeros of the polynomial* $f(z) = \sum_{k=0}^{m} a_k z^k$ *are positive and real and if* p *is an integer,* $1 \leq p \leq m$, *then all the zeros of the polynomial*

$$f_{p\beta}(z) = \sum_{k=0}^{m} \{(-1)^{p-1} k(k-1)\cdots(k-p+1) - \beta\} a_k z^k$$

can be made to lie in any convex sector $S: \theta_1 \leq \arg z \leq \theta_2$ *containing the positive real axis in its interior, provided*

$$|\beta| > p!\, C(m,p) \operatorname{cosec}^p \theta, \quad \text{where } \theta = \min(|\theta_1|, |\theta_2|). \tag{2.5}$$

PROOF. If S is any sector as stipulated in the theorem, then $-\pi < \theta_1 < 0 < \theta_2 < \pi$ and $\theta_2 - \theta_1 = |\theta_1| + |\theta_2| \leq \pi$. For each value of $i = 1, 2$, the arc of circle given by

$$C_i = \{z \in \mathbf{C}: \arg[z/(z-\lambda)] = \theta_i\}$$

is the locus of points at which the line segment from $z = 0$ to $z = \lambda$ subtends a constant angle θ_i (see Theorem (2.1) for the value of λ). Then C_1 and C_2 are the arcs of circles, having end points at $z = 0$ and $z = \lambda$ and lying on opposite sides of the above line segment, with diameters given by $d_1 = |\lambda| \operatorname{cosec} |\theta_1|$ and $d_2 = |\lambda| \operatorname{cosec} |\theta_2|$, respectively. These expressions hold even if $|\theta_1|$ or $|\theta_2|$ exceeds $\pi/2$. Obviously, for each value of $i = 1, 2$, every point $z \in C_i$ satisfies the inequality $|z| \leq d_i$ or $|z| \leq |\lambda|$ according as $|\theta_i| \leq \pi/2$ or $|\theta_i| > \pi/2$. Since $|\theta_1|$ and $|\theta_2|$ cannot exceed $\pi/2$ simultaneously and since $|\lambda| \leq d_i = |\lambda| \operatorname{cosec} |\theta_i|$ for $i = 1, 2$, all the points $z \in C_1 \cup C_2$ satisfy the inequality $|z| \leq d_1$ or $|z| \leq d_2$ according as $|\theta_2| > \pi/2$ or $|\theta_1| > \pi/2$. That is, in either case when $|\theta_1|$ or $|\theta_2|$ exceeds $\pi/2$, we have

$$|z| \leq |\lambda| \operatorname{cosec} \theta \qquad \forall z \in C_1 \cup C_2, \tag{2.6}$$

where $\theta = \min(|\theta_1|, |\theta_2|)$. Obviously, (2.6) still holds when both $|\theta_1|$ and $|\theta_2|$ do not exceed $\pi/2$. To sum up: In all cases, the inequality (2.6) holds for all points in the closed interior of the region bounded by C_1 and C_2.

Now if we take $g(z) = z - \beta$, with β satisfying the inequality (2.5), then all the pth-roots of β (the only zero of $g(z)$) lie on the circle $|z| = |\beta|^{1/p}$, where $|\beta|^{1/p} > |\lambda| \operatorname{cosec} \theta$. But the set of all points lying in the closed exterior of the region bounded by C_1 and C_2 (except the points 0 and λ) is precisely the lune L given by

$$L: \theta_1 \leq \arg[z/(z-\lambda)] \leq \theta_2, \qquad |\theta_1| + |\theta_2| \leq \pi.$$

Therefore, all the pth-roots of the zero of $g(z)$ lie in the lune L. Since all the zeros of $f(z)$ lie in the sector S_0 given by $S_0\colon 0 \leq \arg z \leq 0$, the polynomials f and g satisfy the hypotheses of Theorem (2.1) with $\mu_1 = \mu_2 = \mu = 0$ and $n = 1$. Consequently (since $\theta_1 < 0 < \theta_2$), all the zeros of the polynomial $h(z)$ of Theorem (2.1), with $g(z) = z - \beta$, lie in the sector $S\colon \theta_1 \leq \arg z \leq \theta_2$. Finally, it is easy to see that the polynomial $f_{p\beta}(z)$ of the present theorem is precisely the above polynomial $h(z)$. The proof is now complete.

REMARK (2.4). (I) Since the polynomial $-f_{p\beta}(z)$ in Theorem (2.3) turns out, incidently, to be the polynomial $h_1(z)$ of Lemma (1.1) with $\beta_1 = \beta$, the above theorem can also be proved by using, basically, the same method of proof as above and applying Lemma (1.1) instead of Theorem (2.1).

(II) If $0 < \alpha < \pi$, Theorem (2.3) says that all the zeros of $f_{p\beta}(z)$ lie in the sector

$$H_\alpha\colon \alpha - \pi \leq \arg z \leq \alpha,$$

provided $|\beta| > p!\,C(m,p)\operatorname{cosec}^p \gamma$, where $\gamma = \min(|\alpha - \pi|, |\alpha|)$. If, in particular, $\alpha = \pi/2$, then $H_\alpha = \{z\colon -\pi/2 \leq \arg z \leq \pi/2\}$ and all the zeros of $f_{p\beta}(z)$ lie in the half-plane $\operatorname{Re}(z) \geq 0$, provided $|\beta| > p!\,C(m,p)$.

COROLLARY (2.5). *Under the notations and hypotheses of Theorem* (2.3) *no zero of* $f_{p\beta}(z)$ *can be a nonpositive real number for any value of* β *such that*

$$|\beta| > p!\,C(m,p). \tag{2.7}$$

PROOF. Since $a_0 = f(0) \neq 0$, we see that $f_{p\beta}(0) = -\beta f(0) \neq 0$ for any β satisfying (2.7). It remains only to show that no zero of $f_{p\beta}(z)$ can be a negative real number for any β satisfying (2.7). In order to prove this we suppose, on the contrary, that $f_{p\beta}(\lambda) = 0$ for some negative real number λ and for some β satisfying (2.7). From Remark (2.4)(II) we, therefore, conclude that $\lambda = \operatorname{Re}(\lambda) \geq 0$. Since $\lambda \neq 0$ (already proved), we must have $\lambda > 0$. This contradicts the choice of λ already made, and our corollary is established.

COROLLARY (2.6). *If all the zeros of an* m*th-degree polynomial* $f(z)$ *are real and positive and if* p *is an integer,* $1 \leq p \leq m$, *then*

$$|z|^p |f^{(p)}(z)| \leq p!\,C(m,p)|f(z)| \quad \forall z \in H, \tag{2.8}$$

where $H = \{z \in \mathbf{C}\colon \operatorname{Re}(z) < 0\}$.

PROOF. If we suppose, on the contrary, that

$$|z_0|^p |f^{(p)}(z_0)| > p!\,C(m,p)|f(z_0)| \quad \text{for some } z_0 \in H,$$

then (since $f(z_0) \neq 0$) we must have

$$|(-1)^{p-1} z_0^p f^{(p)}(z_0)/f(z_0)| > p!\,C(m,p).$$

Putting $\beta = (-1)^{p-1} z_0^p f^{(p)}(z_0)/f(z_0)$, we observe that $|\beta| > p!\,C(m,p)$ and that

$$0 = \beta f(z_0) + (-1)^p z_0^p f^{(p)}(z_0) = -f_{p\beta}(z_0)$$

as stated in Remark (2.4)(I). Now Remark (2.4)(II) implies that $\operatorname{Re}(z_0) \geq 0$, contradicting the fact that $z_0 \in H$. The corollary is now proved.

At the end, we prove the following result on the roots of certain partial fraction sums as application of Theorem (2.3).

COROLLARY (2.7). *If, for $j = 1, 2, \ldots, q$, the t_j are positive reals and the m_j are positive integers with sum m, then the roots of the equation*

$$\sum_{j=1}^{q} m_j/(z - t_j) = \beta/z, \tag{2.9}$$

for all complex numbers β lying outside the circle $|z| = m$, lie in the closed half-plane $\mathrm{Re}(z) \geq 0$.

PROOF. If we put

$$f(z) = \prod_{j=1}^{q} (z - t_j)^{m_j} = \sum_{k=0}^{m} a_k z^k$$

and take $\beta \in \mathbf{C}$ such that $|\beta| > m$, then the polynomial $f_{p\beta}(z)$ of Theorem (2.3) for $p = 1$ is given by

$$-f_{1\beta}(z) = \sum_{k=0}^{m} (\beta - k) a_k z^k = \beta f(z) - z f'(z).$$

Since the polynomial f satisfies the hypotheses of Theorem (2.3), with $p = 1$ and $-\theta_1 = \theta_2 = \pi/2$, we conclude (see also Remark (2.4)(II)) that all the zeros of $f_{1\beta}(z)$ lie in the closed half-plane $\mathrm{Re}(z) \geq 0$ (since $|\beta| > m$). Since the left-hand side of equation (2.9) equals $f'(z)/f(z)$, every root of equation (2.9) must be a zero of $f_{1\beta}(z)$. Hence, the corollary has been established.

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