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THE EXTREME POINTS OF Σ

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ABSTRACT. For any compact set in $\mathbf{C}$, with complement Ω which contains ∞ and is connected the class Σ consists of functions $g(z) = z + b_1 z^{-1} + \cdots$ that are univalent in Ω. We prove that $g \in \Sigma$ is an extreme point of Σ if and only if $\mathbf{C} - g(\Omega)$ has zero area.

Introduction. Let A be the complex linear space of all functions analytic in some fixed domain $\Omega \subset \mathbf{C}$, with the topology of locally uniform convergence. A function g in some subset $\mathcal{F} \subset A$ is called an extreme point of $\mathcal{F}$ if

$$g(z) \neq t g_1(z) + (1-t) g_2(z) \tag{1}$$

for all distinct $g_1, g_2 \in \mathcal{F}$ and $t \in (0,1)$.

In the theory of conformal mapping we often assume that $\infty \in \Omega$ and study the class Σ of functions

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^{-n} \tag{2}$$

which are univalent in Ω. This class is a compact subset of A (see for instance [**1**]).

Some time ago Springer [**4**] showed, when $\Omega = \{|z| > 1\}$, that g is an extreme point of Σ when $\mathbf{C} - g(\Omega)$ has zero area. The problem of obtaining the converse result was raised by Schober at a recent conference on Complex Analysis at Lexington, Kentucky (see also [**3**, p. 78 and **2**]). In this paper we obtain the general result:

THEOREM 1. *A function $g(z) \in \Sigma$ is an extreme point of Σ if and only if $\mathbf{C} - g(\Omega)$ has zero area.*

REMARK 1. In fact we prove that $\mathbf{C} - g(\Omega)$ has zero area if and only if g is an extreme point of the closed convex hull of Σ.

The proof is divided into 2 parts. First we use a recent deep result of Nguyen to show that for $g \in \Sigma$ such that $E \equiv \mathbf{C} - g(\Omega)$ has positive area then g is not an extreme point of Σ. The converse result follows from a generalization of Springer's original idea.

1. In [**3**] it is shown that if E has positive area then there is a measure μ, supported on E, such that the Cauchy transform

$$\hat{\mu}(w) = \iint_E \frac{d\mu(\xi)}{(\xi - w)} \tag{3}$$

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is nonconstant, and satisfies a Lipschitz condition, i.e.

$$|\hat{\mu}(w_1) - \hat{\mu}(w_2)| \leq |w_1 - w_2| \tag{4}$$

for any w_1, $w_2 \in \Omega$. From this we obtain

LEMMA 1. *For* $\epsilon < 1$ *the functions*

$$v_j(w) = w + \epsilon(-1)^j \hat{\mu}(w), \qquad j = 1 \text{ or } 2, \tag{5}$$

are univalent in $g(\Omega)$.

This follows immediately from

$$\left|\frac{v_j(w_1) - v_j(w_2)}{(w_1 - w_2)} - 1\right| = \epsilon\left|\frac{\hat{\mu}(w_1) - \hat{\mu}(w_1)}{(w_1 - w_2)}\right| < 1,$$

by (4).

Next we define functions for $j = 1, 2$:

$$g_j(z) = v_j(g(z)).$$

From Lemma 1 we see that g_j is univalent and in fact, by (3) and (5), $g_j \in \Sigma$. Also by (5), as $\hat{\mu}$ is nonzero, $g_1 \neq g_2$ and $g = \frac{1}{2}g_1 + \frac{1}{2}g_2$. Consequently g is not an extreme point of Σ.

2. We now prove the converse result, i.e. suppose that $\mathbf{C} - g(\Omega)$ has zero area.

LEMMA 2. *There is a union* Γ *of a countable number of Jordan arcs such that*
(i) $\mathbf{C} - (D \cup \Gamma)$ *is simply connected.*
(ii) $\text{Area}(\Gamma) = 0$.

There is a sequence $D_n \downarrow D$ such that D_n is the union of a finite number of disjoint sets $D_{n,j}$ which have smooth Jordan curves as their boundaries.

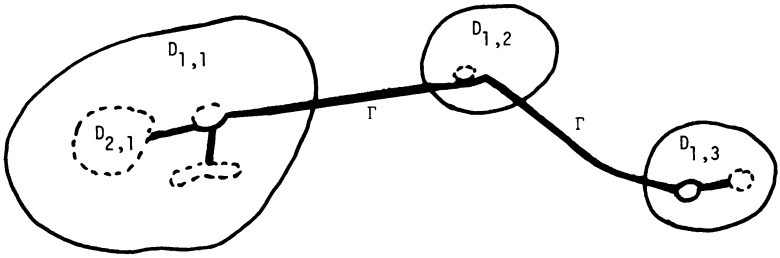

FIGURE 1

Suppose that $D_{n,n_1} \cdots D_{n,n_p}$ lie in $D_{n-1,k}$. Then clearly $D_{n,n_1} \cdots D_{n,n_p}$ may be joined by $(p-1)$ arcs in $D_{n-1,k} - (D_{n,n_1} \cdots D_{n,n_p})$. Then Γ is obtained by joining this collection of arcs (in the obvious way). One does this in such a manner as to ensure that $D \cup \Gamma$ is closed and $\mathbf{C} - (D \cup \Gamma)$ is connected. Also one makes sure that Γ is bounded. Clearly $\text{Area}(\Gamma) = 0$.

Now let $\Omega_0 = \mathbf{C} - (D \cup \Gamma)$. We map Ω_0 by a univalent function

$$F(z) = z + \sum_{n=0}^{\infty} \gamma_n z^{-n}$$

onto the exterior Δ of a disk $\{|x| < R\}$. Let $\mathcal{H}$ be the class of functions

$$h(x) = a_{-1}x + \sum_{n=0}^{\infty} a_n x^{-n}$$

analytic in Δ and satisfying

$$||h||_2^2 \equiv |a_{-1}|^2 R^{-2} + |a_0|^2 + \sum_{n=1}^{\infty} n|a_n|^2 R^{2n} < \infty.$$

Then, with the obvious definitions, $\mathcal{H}$ is a Hilbert space.

Next we embed Σ, or rather the closed convex hull of Σ, in $\mathcal{H}$. The transformation

$$g \to h \equiv g(F^{-1}(x)) \tag{5}$$

provides a linear mapping of Σ into $\mathcal{H}$. To see this first note that $g(F^{-1})$ is univalent in Δ and has normalisation

$$h(x) = x + 0(1)$$

near ∞. The area theorem [**3**, p. 176] shows that

$$\sum_{n=1}^{\infty} n|a_n|^2 R^{2n} \leq R^{-2},$$

i.e. Σ is embedded in a ball B of radius R^{-1} and centre $z + \gamma_0$. Furthermore, by the area theorem, h is on the boundary of B if and only if $\mathbf{C} - h(\Delta)$ has zero area.

LEMMA 3. *Suppose that $g \in \Sigma$ and $\mathbf{C} - g(\Omega)$ has zero area. Then, with h defined by* (5), $\mathbf{C} - h(\Delta)$ *has zero area.*

In Lemma 2 the D_n may be chosen so that

$$\text{Area}(\mathbf{C} - g(\mathbf{C} - D_n)) \downarrow 0.$$

Then Lemma 3 follows from

$$\begin{aligned} \text{Area}(\mathbf{C} - h(\Delta)) &= \text{Area}(\mathbf{C} - g(\mathbf{C} - (D \cup \Gamma))) \\ &\leq \text{Area}(\mathbf{C} - g(\mathbf{C} - D_n)) + \text{Area}(g\{(\mathbf{C} - D_n) \cap \Gamma\}) \\ &= \text{Area}(\mathbf{C} - g(\mathbf{C} - D_n)). \end{aligned}$$

Finally we prove the result wanted. Assume that $g \in \Sigma$ and $\mathbf{C} - g(\Omega)$ has zero area, but g is not an extreme point of Σ. Thus there exist g_1, $g_2 \in \Sigma$, $t \in (0,1)$ such that $g_1 \neq g_2$ and

$$g = tg_1 + (1-t)g_2. \tag{6}$$

The transformation into $\mathcal{H}$ gives

$$h = th_1 + (1-t)h_2, \tag{7}$$

with $h_j \in B$. Lemma 3 shows that $\text{Area}(\mathbf{C} - h(\Delta)) = 0$. Consequently h is on the boundary of a ball B in Hilbert space. In Hilbert space each boundary point of a ball is also an extreme point of that ball. Hence h is an extreme point of B and thus $h_1 = h_2$. However, this implies that $g_1 = g_2$ which is a contradiction.

This completes the proof of the theorem.

3. We conclude by examining the trivial cases, i.e. when does Σ only consist of extreme points? The above theorem shows that every g maps Ω onto a set whose complement has zero area. Then a result of Ahlfors and Beurling (Theorem 5 of [**1**]) shows that D is a null set with respect to the Dirichlet class on Ω, i.e. those f analytic on Ω such that $\iint_\Omega |f'|^2 \, dA < \infty$. Then we use Theorem 6 (of [**1**]) to show that linear functions are the only univalent functions defined on Ω. Thus Σ consists just of the function $g(z) \equiv z$. The converse is true too. Hence the trivial cases for Σ are all the same case, i.e. when D is a null set for the Dirichlet class.

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REFERENCES

1. L. V. Ahlfors and A. Beurling, *Conformal invariants and function-theoretic null sets*, Acta. Math. **83** (1950), 103–129.
2. W. E. Kirwan and G. Schober, *On extreme points and support points, for some families of univalent functions*, Duke Math. J. **42** (1976), 285–296.
3. Nguyen Xuan Uy, *Removable sets of analytic functions satisfying a Lipschitz condition*, Ark. Math. **17** (1979), 14–27.
4. G. Schober, *Univalent functions*, Lecture Notes in Math., vol. 478, Springer-Verlag, Berlin and New York, 1975.
5. G. Springer, *Extreme Punkte der Konvexen Hülle schlichter Funktionen*, Math. Ann. **129** (1955), 230–232.

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