

## THE EXTREME POINTS OF $\Sigma$

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**ABSTRACT.** For any compact set in  $\mathbb{C}$ , with complement  $\Omega$  which contains  $\infty$  and is connected the class  $\Sigma$  consists of functions  $g(z) = z + b_1 z^{-1} + \dots$  that are univalent in  $\Omega$ . We prove that  $g \in \Sigma$  is an extreme point of  $\Sigma$  if and only if  $\mathbb{C} - g(\Omega)$  has zero area.

**Introduction.** Let  $A$  be the complex linear space of all functions analytic in some fixed domain  $\Omega \subset \mathbb{C}$ , with the topology of locally uniform convergence. A function  $g$  in some subset  $\mathcal{F} \subset A$  is called an extreme point of  $\mathcal{F}$  if

$$(1) \quad g(z) \neq tg_1(z) + (1-t)g_2(z)$$

for all distinct  $g_1, g_2 \in \mathcal{F}$  and  $t \in (0, 1)$ .

In the theory of conformal mapping we often assume that  $\infty \in \Omega$  and study the class  $\Sigma$  of functions

$$(2) \quad g(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$$

which are univalent in  $\Omega$ . This class is a compact subset of  $A$  (see for instance [1]).

Some time ago Springer [4] showed, when  $\Omega = \{|z| > 1\}$ , that  $g$  is an extreme point of  $\Sigma$  when  $\mathbb{C} - g(\Omega)$  has zero area. The problem of obtaining the converse result was raised by Schober at a recent conference on Complex Analysis at Lexington, Kentucky (see also [3, p. 78 and 2]). In this paper we obtain the general result:

**THEOREM 1.** *A function  $g(z) \in \Sigma$  is an extreme point of  $\Sigma$  if and only if  $\mathbb{C} - g(\Omega)$  has zero area.*

**REMARK 1.** In fact we prove that  $\mathbb{C} - g(\Omega)$  has zero area if and only if  $g$  is an extreme point of the closed convex hull of  $\Sigma$ .

The proof is divided into 2 parts. First we use a recent deep result of Nguyen to show that for  $g \in \Sigma$  such that  $E \equiv \mathbb{C} - g(\Omega)$  has positive area then  $g$  is not an extreme point of  $\Sigma$ . The converse result follows from a generalization of Springer's original idea.

1. In [3] it is shown that if  $E$  has positive area then there is a measure  $\mu$ , supported on  $E$ , such that the Cauchy transform

$$(3) \quad \hat{\mu}(w) = \iint_E \frac{d\mu(\xi)}{(\xi - w)}$$

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Received by the editors July 10, 1981.

1980 *Mathematics Subject Classification.* Primary 30C55.

<sup>1</sup>The author was partially funded under NSF grant no. 81-0380.

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0002-9939/81/0000-0306/\$02.00

is nonconstant, and satisfies a Lipschitz condition, i.e.

$$(4) \quad |\hat{\mu}(w_1) - \hat{\mu}(w_2)| \leq |w_1 - w_2|$$

for any  $w_1, w_2 \in \Omega$ . From this we obtain

LEMMA 1. For  $\epsilon < 1$  the functions

$$(5) \quad v_j(w) = w + \epsilon(-1)^j \hat{\mu}(w), \quad j = 1 \text{ or } 2,$$

are univalent in  $g(\Omega)$ .

This follows immediately from

$$\left| \frac{v_j(w_1) - v_j(w_2)}{(w_1 - w_2)} - 1 \right| = \epsilon \left| \frac{\hat{\mu}(w_1) - \hat{\mu}(w_2)}{(w_1 - w_2)} \right| < 1,$$

by (4).

Next we define functions for  $j = 1, 2$ :

$$g_j(z) = v_j(g(z)).$$

From Lemma 1 we see that  $g_j$  is univalent and in fact, by (3) and (5),  $g_j \in \Sigma$ . Also by (5), as  $\hat{\mu}$  is nonzero,  $g_1 \neq g_2$  and  $g = \frac{1}{2}g_1 + \frac{1}{2}g_2$ . Consequently  $g$  is not an extreme point of  $\Sigma$ .

2. We now prove the converse result, i.e. suppose that  $C - g(\Omega)$  has zero area.

LEMMA 2. There is a union  $\Gamma$  of a countable number of Jordan arcs such that

- (i)  $C - (D \cup \Gamma)$  is simply connected.
- (ii)  $\text{Area}(\Gamma) = 0$ .

There is a sequence  $D_n \downarrow D$  such that  $D_n$  is the union of a finite number of disjoint sets  $D_{n,j}$  which have smooth Jordan curves as their boundaries.

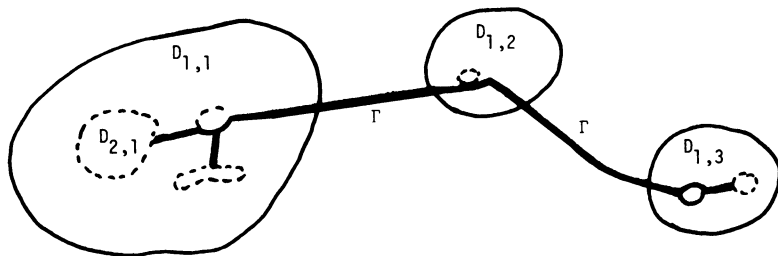


FIGURE 1

Suppose that  $D_{n,n_1} \cdots D_{n,n_p}$  lie in  $D_{n-1,k}$ . Then clearly  $D_{n,n_1} \cdots D_{n,n_p}$  may be joined by  $(p - 1)$  arcs in  $D_{n-1,k} - (D_{n,n_1} \cdots D_{n,n_p})$ . Then  $\Gamma$  is obtained by joining this collection of arcs (in the obvious way). One does this in such a manner as to ensure that  $D \cup \Gamma$  is closed and  $C - (D \cup \Gamma)$  is connected. Also one makes sure that  $\Gamma$  is bounded. Clearly  $\text{Area}(\Gamma) = 0$ .

Now let  $\Omega_0 = C - (D \cup \Gamma)$ . We map  $\Omega_0$  by a univalent function

$$F(z) = z + \sum_{n=0}^{\infty} \gamma_n z^{-n}$$

onto the exterior  $\Delta$  of a disk  $\{|x| < R\}$ . Let  $\mathcal{H}$  be the class of functions

$$h(x) = a_{-1}x + \sum_{n=0}^{\infty} a_n x^{-n}$$

analytic in  $\Delta$  and satisfying

$$\|h\|_2^2 \equiv |a_{-1}|^2 R^{-2} + |a_0|^2 + \sum_{n=1}^{\infty} n|a_n|^2 R^{2n} < \infty.$$

Then, with the obvious definitions,  $\mathcal{H}$  is a Hilbert space.

Next we embed  $\Sigma$ , or rather the closed convex hull of  $\Sigma$ , in  $\mathcal{H}$ . The transformation

$$(5) \quad g \rightarrow h \equiv g(F^{-1}(x))$$

provides a linear mapping of  $\Sigma$  into  $\mathcal{H}$ . To see this first note that  $g(F^{-1})$  is univalent in  $\Delta$  and has normalisation

$$h(x) = x + O(1)$$

near  $\infty$ . The area theorem [3, p. 176] shows that

$$\sum_{n=1}^{\infty} n|a_n|^2 R^{2n} \leq R^{-2},$$

i.e.  $\Sigma$  is embedded in a ball  $B$  of radius  $R^{-1}$  and centre  $z + \gamma_0$ . Furthermore, by the area theorem,  $h$  is on the boundary of  $B$  if and only if  $\mathbf{C} - h(\Delta)$  has zero area.

LEMMA 3. *Suppose that  $g \in \Sigma$  and  $\mathbf{C} - g(\Omega)$  has zero area. Then, with  $h$  defined by (5),  $\mathbf{C} - h(\Delta)$  has zero area.*

In Lemma 2 the  $D_n$  may be chosen so that

$$\text{Area}(\mathbf{C} - g(\mathbf{C} - D_n)) \downarrow 0.$$

Then Lemma 3 follows from

$$\begin{aligned} \text{Area}(\mathbf{C} - h(\Delta)) &= \text{Area}(\mathbf{C} - g(\mathbf{C} - (D \cup \Gamma))) \\ &\leq \text{Area}(\mathbf{C} - g(\mathbf{C} - D_n)) + \text{Area}(g\{(\mathbf{C} - D_n) \cap \Gamma\}) \\ &= \text{Area}(\mathbf{C} - g(\mathbf{C} - D_n)). \end{aligned}$$

Finally we prove the result wanted. Assume that  $g \in \Sigma$  and  $\mathbf{C} - g(\Omega)$  has zero area, but  $g$  is not an extreme point of  $\Sigma$ . Thus there exist  $g_1, g_2 \in \Sigma$ ,  $t \in (0, 1)$  such that  $g_1 \neq g_2$  and

$$(6) \quad g = tg_1 + (1 - t)g_2.$$

The transformation into  $\mathcal{H}$  gives

$$(7) \quad h = th_1 + (1 - t)h_2,$$

with  $h_j \in B$ . Lemma 3 shows that  $\text{Area}(\mathbf{C} - h(\Delta)) = 0$ . Consequently  $h$  is on the boundary of a ball  $B$  in Hilbert space. In Hilbert space each boundary point of a ball is also an extreme point of that ball. Hence  $h$  is an extreme point of  $B$  and thus  $h_1 = h_2$ . However, this implies that  $g_1 = g_2$  which is a contradiction.

This completes the proof of the theorem.

3. We conclude by examining the trivial cases, i.e. when does  $\Sigma$  only consist of extreme points? The above theorem shows that every  $g$  maps  $\Omega$  onto a set whose complement has zero area. Then a result of Ahlfors and Beurling (Theorem 5 of [1]) shows that  $D$  is a null set with respect to the Dirichlet class on  $\Omega$ , i.e. those  $f$  analytic on  $\Omega$  such that  $\iint_{\Omega} |f'|^2 dA < \infty$ . Then we use Theorem 6 (of [1]) to show that linear functions are the only univalent functions defined on  $\Omega$ . Thus  $\Sigma$  consists just of the function  $g(z) \equiv z$ . The converse is true too. Hence the trivial cases for  $\Sigma$  are all the same case, i.e. when  $D$  is a null set for the Dirichlet class.

The author acknowledges discussions with C. FitzGerald, G. Schober and C. Pommerenke.

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