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RECURRENCE RELATIONS FOR MULTIVARIATE B-SPLINES[1]

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ABSTRACT. We prove recurrence relations for a general class of multivariate B-splines, obtained as 'projections' of convex polyhedra. Our results are simple consequences of Stokes' theorem and include, as special cases, the recurrence relations for the standard multivariate simplicial B-spline.

We wish to point out what, in hindsight, seems obvious, namely that the recurrence relations for multivariate B-splines established by C. A. Micchelli [**19**] and reproved in various different ways by W. Dahmen [**6**], C. A. Micchelli [**20**], K. Höllig [**15**] and H. Hakopian [**14**] (and perhaps others) are special cases of more general and very simple recurrence relations which are a simple consequence of Stokes' theorem.

To recall, following the lead of I. J. Schoenberg [**21**], the multivariate B-spline $M(\cdot|x_0,\dots,x_n)$ was defined in [**1**] by the rule

$$M(x|x_0,\dots,x_n) := \frac{\operatorname{vol}_{n-m}(\{z\in R^n : Pz = x\}\cap \operatorname{conv}\{x_0,\dots,x_n\})}{\operatorname{vol}_n \operatorname{conv}\{x_0,\dots,x_n\}}, \qquad x\in R^m,$$

with $x_0,\dots,x_n$ points in R^n and $\operatorname{conv}\{x_0,\dots,x_n\}$ their convex hull, with $\operatorname{vol}_k(K)$ the k-dimensional volume of the set K, and

$$P\colon R^n \to R^m\colon z\mapsto (z(i))_{i=1}^m.$$

Such a B-spline is a nonnegative piecewise polynomial function of degree at most $n-m$, its support is $\operatorname{conv}\{Px_0,\dots,Px_n\}$, and it is in C^{n-m-1} as long as the "knots" $Px_0,\dots,Px_n$ are in general position.

It was hoped that these functions could be made to play the same basic role in the analysis and use of smooth multivariate piecewise polynomial functions that their much older univariate version (introduced by Curry and Schoenberg [**4–5**]) had assumed in the univariate spline theory. These hopes have already borne some fruit; see Micchelli [**20**], Dahmen [**7–9**], Dahmen and Micchelli [**10–12**], Goodman and Lee [**13**], Höllig [**14**]. The first step in this development was taken by C. A. Micchelli [**19**] who proved the following.

THEOREM 1 (C. A. MICCHELLI). (i) *If* $z=\sum\lambda_i Px_i$ *with* $\sum\lambda_i = 0$, *then*

$$D_z M(\cdot|x_0,\dots,x_n) = n\sum \lambda_i M(\cdot|x_0,\dots,x_{i-1},x_{i+1},\dots,x_n).$$

(ii) *If* $x=\sum\lambda_i Px_i$ *with* $\sum\lambda_i = 1$, *then*

$$(n-m)M(x|x_0,\dots,x_n) = n\sum \lambda_i M(x|x_0,\dots,x_{i-1},x_{i+1},\dots,x_n).$$

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Here, $D_z f := \sum z(i) D_i f$, with $D_i f$ the partial derivative of f with respect to its ith argument. Further, the equalities asserted in the theorem must in general be interpreted in the sense of distributions. In this connection, Micchelli's starting point was the observation that

$$\int_{R^m} M(\cdot|x_0, \dots, x_n)\phi$$
$$= n! \int_0^1 \cdots \int_0^{t_{n-1}} (\phi \circ P)(x_0 + t_1(x_1 - x_0) + \cdots + t_n(x_n - x_{n-1}))\, dt_n \cdots dt_1.$$

These integrals play a crucial role in Kergin interpolation [**17–19**]. They also appear in the Hermite-Genocchi formula for the nth divided difference.

Consider now, more generally, a polyhedral convex body B in R^n, whose boundary ∂B is the essentially disjoint union of finitely many $(n-1)$-dimensional convex bodies B_i with corresponding outward normal n_i. Let M and M_i denote the corresponding distributions on R^m defined by the rule

$$M\phi := \int_B \phi \circ P, \quad M_i\phi := \int_{B_i} \phi \circ P, \quad \text{all test functions } \phi.$$

Here, $\int_K$ denotes the k-dimensional integral over the convex set K in case K spans a k-dimensional flat.

THEOREM 2. (i) $D_{Pz}M = -\sum \langle x|n_i\rangle M_i$, *all* $z \in R^n$.
(ii) $(n-m)M(Pz) = \sum \langle b_i - z|n_i\rangle M_i(Pz)$, *all* $z \in R^n$.

Here, b_i stands for an arbitrary point in the flat spanned by B_i, hence the coefficient $\langle b_i - z|n_i\rangle$ is simply the signed distance of z from that flat.

The proof of (i) is immediate:

$$(D_{Pz}M)\phi = -\int_B (D_{Pz}\phi) \circ P = -\int_B D_z(\phi \circ P) = -\int_{\partial B} \langle z|n\rangle \phi \circ P.$$

As to (ii), we follow Hakopian [**14**] who derives Theorem 1(ii) from the following B-spline identity:

$$(D - D_{x_i})M(\cdot|x_0, \dots, x_n)$$
$$= (n-m)M(\cdot|x_0, \dots, x_n) - nM(\cdot|x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

Here, D stands for the differential operator given by the rule

$$(Df)(x) := \sum_{j=1}^{k} x(j)(D_j f)(x)$$

for a function f of k variables.

Correspondingly, we prove

(iii) $DM = (n-m)M - \sum \langle b_i|n_i\rangle M_i$

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as follows:

$$\begin{aligned}(DM)\phi &= -\int_B \sum_{j=1}^m [D_j(x(j)\phi)](Px)\,dx = -mM\phi - \int_B \sum_{j=1}^m [x(j)D_j\phi](Px)\,dx \\ &= -mM\phi - \int_B \sum_{j=1}^n x(j)D_j(\phi\circ P)(x)\,dx \\ &= (n-m)M\phi - \int_B \sum_{j=1}^n D_j[x(j)(\phi\circ P)](x)\,dx \\ &= (n-m)M\phi - \sum \int_{B_i} \langle x|n_i\rangle(\phi\circ P)(x)\,dx\end{aligned}$$

and this proves (iii) since $\langle\cdot|n_i\rangle$ is constant on B_i.

Now, to prove (ii), conclude from (i) and (iii) that, for any z with $Pz = x$,

$$\begin{aligned}0 &= (D - D_{Pz})M(x) \\ &= (n-m)M(x) - \sum\langle b_i|n_i\rangle M_i(x) + \sum\langle z|n_i\rangle M_i(x).\end{aligned}$$

REMARKS. (a) The convexity assumption is sufficient for the intended application but could, of course, be relaxed.

(b) Repeated application of Theorem 2(i) shows that M is a piecewise polynomial of degree at most $n-m$, with possible discontinuities only across convex sets of dimension $m-1$ of the form $P[F]$, with F a face of B. Precisely, $M \in C^{n-d-2}$ with d the greatest integer with the property that a d-dimensional face of B is projected by P into an $(m-1)$-dimensional set.

(c) This study was motivated by the realization that some standard finite elements could be obtained as such 'projections' of simple geometric bodies and by the hope that, by using bodies other than simplices, the resulting piecewise polynomial functions M might be simpler and conform more easily to standard meshes. First results along these lines are contained in [**2** and **3**].

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