CLEAR VISIBILITY AND THE DIMENSION OF KERNELS OF STARSHAPED SETS

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ABSTRACT. This paper will use the concept of clearly visible to obtain a Krasnosel'skii-type theorem for the dimension of the kernel of a starshaped set, and the following result will be proved: For each \( k \) and \( n \), \( 1 \leq k \leq n \), let 
\[
f(n, n) = n + 1 \quad \text{and} \quad f(n, k) = 2n \quad \text{if} \quad 1 \leq k < n - 1.
\]
Let \( S \) be a nonempty compact set in \( \mathbb{R}^n \). Then for a \( k \) with \( 1 \leq k \leq n \), \( \dim \ker S \geq k \) if and only if every \( f(n, k) \) points of \( \partial S \) are clearly visible from a common \( k \)-dimensional subset of \( S \). If \( k = 1 \) or \( k = n \), the result is best possible. Moreover, if \( S \) is a compact, connected, nonconvex set in \( \mathbb{R}^2 \), then \( \partial S \) may be replaced by \( \text{lnc} S \) in the theorem.

1. Introduction. We begin with some definitions from [1]. Let \( S \) be a compact set in \( \mathbb{R}^n \). A point \( s \) in \( S \) is said to be a point of local convexity of \( S \) if and only if there is some neighborhood \( N \) of \( s \) such that \( N \cap S \) is convex. If \( S \) fails to be locally convex at \( q \) in \( S \), then \( q \) is called a point of local nonconvexity (lnc) point of \( S \). For points \( x \) and \( y \) in \( S \), we say \( x \) sees \( y \) via \( S \) (\( x \) is visible from \( y \) via \( S \)) if and only if the segment \( [x, y] \) lies in \( S \). Similarly, \( x \) is clearly visible from \( y \) via \( S \) if and only if there is some neighborhood \( N \) of \( x \) such that \( y \) sees each point of \( N \cap S \) via \( S \). Finally, set \( S \) is starshaped if and only if there is some point \( p \) in \( S \) such that \( p \) sees each point of \( S \) via \( S \), and the set of all such points \( p \) is called the (convex) kernel of \( S \), denoted \( \ker S \).

A theorem of Krasnosel'skii [6] states that if \( S \) is a nonempty compact set in \( \mathbb{R}^n \), then \( S \) is starshaped if and only if every \( n + 1 \) points of \( S \) are visible from a common point of \( S \). (A stronger result may be obtained by replacing points of \( S \) with boundary points of \( S \).) In [2], an analogue of the Krasnosel'skii theorem was proved for the dimension of this kernel: For each \( k \) and \( n \), \( 1 \leq k \leq n \), let 
\[
f(n, n) = n + 1 \quad \text{and} \quad f(n, k) = 2n \quad \text{if} \quad 1 \leq k < n - 1.
\]
If \( S \) is a compact set in \( \mathbb{R}^n \), then \( \ker S \) has dimension at least \( k \) if and only if for some \( \varepsilon > 0 \), every \( f(n, k) \) points of \( S \) see via \( S \) a common \( k \)-dimensional \( \varepsilon \)-neighborhood. Unfortunately, the uniform lower bound \( \varepsilon \) is necessary by an example in [3].

In order to obtain a theorem for the dimension of the kernel independent of this cumbersome \( \varepsilon \)-bound, we turn to the notion of clearly visible, previously appearing in work by Stavrakas [8] and Falconer [4]. In [1], analogues of the Krasnosel'skii theorem were obtained by replacing the concept of visible with that of clearly visible and by replacing points of \( S \) with lnc points of \( S \). A similar approach proves helpful here, and we have the following result: For \( f(n, k) \) defined above, \( 1 \leq k \leq n \), and for \( S \) a nonempty compact set in \( \mathbb{R}^n \), \( \ker S \) has dimension at least \( k \) if and only if every \( f(n, k) \) boundary points of \( S \) are clearly visible from a common \( k \)-dimensional
subset of $S$. If $k = 1$ or $k = n$, the result is best possible. Moreover, if $S$ is a compact, connected, nonconvex set in $R^2$, then the boundary of $S$ may be replaced by the lnc points of $S$.

The following terminology will be used: conv $S$, bdry $S$, and ker $S$ will denote the convex hull, boundary, and kernel, respectively, for set $S$. Lnc $S$ will be the set of points of local nonconvexity of $S$, and if $S$ is convex, dim $S$ will be the dimension of $S$. Finally, $c$ will represent the Hausdorff metric defined on the collection of compact, convex subsets of $R^n$. The reader is referred to Valentine [9] for a discussion of these concepts and to Nadler [7] for further information on the Hausdorff metric.

2. The results. The first lemma is a variation of a result in [2].

**LEMMA 1.** For each $k$ and $n$, $1 < k < n$, let $f(n, n) = n + 1$ and $f(n, k) = 2n$ if $1 < k < n - 1$. Let $B$ be a uniformly bounded collection of compact convex sets in $R^n$ which is closed with respect to the Hausdorff metric. Then for a $k$ with $1 < k < n$, $\dim \bigcap \{B : B \in B\} \geq k$ if and only if every $f(n, k)$ members of $B$ contain a common $k$-dimensional set.

**PROOF.** We need only establish the sufficiency of the condition. Observe that if every $f(n, k)$ members of $B$ contain a common $k$-dimensional set, then by Helly's theorem in $R^n$, $\bigcap \{B : B \in B\} \neq \emptyset$. If $k = n$, suppose on the contrary that $\dim \bigcap \{B : B \in B\} < n$. Then by a theorem of Falconer [4, Theorem 1], there exist $r$ sets $B_1, \ldots, B_r$ in $B$ such that $\dim (B_1 \cap \cdots \cap B_r) = q$, where $q < n$ and $r \leq 2(n - q)$. However, by a result of Katchalski [5, Theorem a], every finite subfamily of $B$ has an $n$-dimensional intersection. We have a contradiction, our supposition is false, and the result is established when $k = n$.

To establish the result for arbitrary $k$, $1 < k < n$, we adapt an argument from [2, Lemma]. The inductive proof is sketched below. If $n = 1$, then $k = 1$, and we have our result. Assume the result is true for natural numbers less than $n$, $2 \leq n$. If $k = n$, again the proof is immediate, so assume $1 < k < n$. If $\dim \bigcap \{B : B \in B\} = n$, there is nothing to prove, so suppose that $\dim \bigcap \{B : B \in B\} < n$. Then by Falconer's result, there are $r$ sets $B_1, \ldots, B_r$ in $B$ with $\dim (B_1 \cap \cdots \cap B_r) = q$, for some $q < n$ and $r \leq 2(n - q)$. Using Katchalski's theorem, $q \geq k$.

Let $G$ denote the $q$-dimensional flat in $R^n$ determined by $B_1 \cap \cdots \cap B_r$. Using our hypothesis and the fact that $1 \leq k \leq q < n$, it is not hard to show that every $2q$ sets in $B$ meet in a $k$-dimensional set in $G$. Hence every $f(q, k)$ members of $\{B \cap G : B \in B\}$ meet in a $k$-dimensional set, and by our induction hypothesis, $\dim \bigcap \{B \cap G : B \in B\} \geq k$, finishing the induction and completing the proof of the lemma.

The following definitions will be helpful.

**DEFINITION.** Let $S$ be a compact set in $R^n$ and let $q \in \text{bdry} S$. We define

$$S_q = \{x : [x, q] \subseteq S\},$$

$$A_q = \{x : q \text{ is clearly visible from } x \text{ via } S\},$$

and

$$C_q = \bigcap \{H : H \text{ a closed halfspace with } A_q \subseteq H \text{ and } q \in \text{bdry } H\}.$$  

Using these definitions, we establish two more lemmas.
Lemma 2. Let $S$ be a compact set in $\mathbb{R}^n$. If $B$ is in the $\sigma$-closure of $\mathcal{C} \equiv \{C_q \cap \text{conv } S : q \in \text{bdry } S\}$, then $B$ contains some set $A_q$.

Proof. Let $\{C_{q_n} \cap \text{conv } S : n \geq 1\}$ be a sequence in $\mathcal{C}$ converging to $B$. Then since $S$ is bounded, some subsequence $\{q'_{n}\}$ of $\{q_n\}$ converges to a point $q$, and clearly $q \in \text{bdry } S$. Since $\{C_{q'} \cap \text{conv } S\}$ also converges to $B$, for convenience of notation we may assume that $\{q_n\}$ converges to $q$. We assert that $A_q \subseteq B$ for this particular $q$. Let $x \in A_q$. Then $q$ is clearly visible via $S$ from $x$, so for some neighborhood $N$ of $q$, $x$ sees via $S$ each point of $N \cap S$. There is some integer $M$ such that whenever $n > M$, $q_n \in N$. For $n > M$, $q_n$ is clearly visible via $S$ from $x$, and so $x \in A_{q_n} \subseteq C_{q_n} \cap \text{conv } S$ for $n > M$. Since $\{C_{q_n} \cap \text{conv } S : n > M\}$ converges to $B$, $x \in B$. Thus $A_q \subseteq B$ and the lemma is established.

Corollary. Let $S$ be a compact set in $\mathbb{R}^n$, and let $k$ and $j$ be fixed, $0 < k < n$. If every $j$ sets in $\{A_q : q \in \text{bdry } S\}$ have at least a $k$-dimensional intersection, then every $j$ sets in the $\sigma$-closure of $\mathcal{C} \equiv \{C_q \cap \text{conv } S : q \in \text{bdry } S\}$ have at least a $k$-dimensional intersection.

Lemma 3. Let $S$ be a compact set in $\mathbb{R}^n$, and let $\mathcal{C} \equiv \{C_q \cap \text{conv } S : q \in \text{bdry } S\}$. Then $\ker S = \bigcap \{B : B \in \text{the } \sigma$-closure of $\mathcal{C}\}$.

Proof. For convenience, let $\mathcal{B}$ denote the $\sigma$-closure of $\mathcal{C}$. Using Lemma 2, clearly $\ker S \subseteq \bigcap \{A_q : q \in \text{bdry } S\} \subseteq \bigcap \{B : B \in \mathcal{B}\}$, so we need to establish only the reverse inclusion. Let $x$ belong to $\mathbb{R}^n \sim \ker S$. By Krasnosel'skii's Lemma [9, Lemma 6.2], there is a $z$ in $\text{bdry } S$ and a closed halfspace $H$ such that $z \in \text{bdry } H$, $S_z \subseteq H$, and $x \notin H$. Since $A_z \subseteq S_z$, we have $A_z \subseteq H$, $x \notin C_z$, and $x \notin \bigcap \{B : B \in \mathcal{B}\}$. Thus the lemma is proved.

Theorem 1. For each $k$ and $n$, $1 \leq k \leq n$, let $f(n, n) = n + 1$ and $f(n, k) = 2n$ if $1 \leq k \leq n - 1$. Let $S$ be a nonempty compact set in $\mathbb{R}^n$. Then for a $k$ with $1 \leq k \leq n$, $\dim \ker S \geq k$ if and only if every $f(n, k)$ points of $\text{bdry } S$ are clearly visible from a common $k$-dimensional subset of $S$. If $k = 1$ or $k = n$, the result is best possible.

Proof. The necessity of the condition is obvious. To establish its sufficiency, assume that every $f(n, k)$ points of $\text{bdry } S$ are clearly visible from a common $k$-dimensional subset of $S$. Using our hypothesis, every $f(n, k)$ members of $\{A_q : q \in \text{bdry } S\}$ have at least a $k$-dimensional intersection. Therefore, by the corollary to Lemma 2, every $f(n, k)$ sets in the $\sigma$-closure of $\mathcal{C} \equiv \{C_q \cap \text{conv } S : q \in \text{bdry } S\}$ have at least a $k$-dimensional intersection. Standard arguments involving the Hausdorff metric [7, 9] show that $\mathcal{B}$ is a uniformly bounded collection of compact convex sets in $\mathbb{R}^n$. Hence by Lemma 1, $\dim \bigcap \{B : B \in \mathcal{B}\} \geq k$. Using Lemma 3, $\ker S = \bigcap \{B : B \in \mathcal{B}\}$, so $\dim \ker S \geq k$, and the theorem is proved.

To see that the number $f(n, 1) = 2n$ is best possible, consider the following example, adapted from a construction by Katchalski [5, Theorem b, Case 1].

Example 1. For convenience of notation, use the component representation $(z_1, \ldots, z_n)$ for a point in $\mathbb{R}^n$, and let $D$ denote the closed unit ball centered at the origin. Construct a compact set $S$ in $\mathbb{R}^n$ such that $\text{Ine } S$ is a union of $2n$ disjoint convex sets $K_1, \ldots, K_{2n}$, each of dimension $n - 2$, and having the following properties:

(1) For $x, y$ in $K_i$, $C_x = C_y$.
(2) For $x$ in $K_i$, $1 \leq i \leq n$, $D \cap \{(z_1, \ldots, z_n): z_i \geq 0\} \subseteq A_x$ and $A_x \cap \{(z_1, \ldots, z_n): z_i < 0\} = \emptyset$.

For $x$ in $K_{i+n}$, $1 \leq i \leq n$, $D \cap \{(z_1, \ldots, z_n): z_i \leq 0\} \subseteq A_x$ and $A_x \cap \{(z_1, \ldots, z_n): z_i > 0\} = \emptyset$.

(3) $K_i \cap K_{i+n} \subseteq \{(z_1, \ldots, z_n): z_i = 0\}, 1 \leq i \leq n$.

Then for $q_i \in K_i$, $1 \leq i \leq 2n$, the corresponding sets $A_{q_i}$ have only the origin in common. However, every $2n - 1$ of the sets in $\{A_q: q \in \text{bdry } S\}$ share a common interval in $D$. Thus the number $f(n, 1) = 2n$ is best possible.

Figure 1 below illustrates the example in $R^2$.

Similarly, our second example shows that the number $f(n, n) = n + 1$ is best.

EXAMPLE 2. Let $S$ be a compact set in $R^n$, with $\text{inc } S$ a union of $n + 1$ disjoint convex sets $K_1, \ldots, K_{n+1}$, each of dimension $n - 2$, and satisfying the following properties:

(1) For $x, y \in K_i$, $C_x = C_y$.

(2) Every $n$ members of $\{C_{q_i}: q_i \in K_i, 1 \leq i \leq n+1\}$ meet in an $n$-dimensional set in $S$, yet $\bigcap \{C_{q_i}: q_i \in K_i, 1 \leq i \leq n\} = \emptyset$.

This may be done so that every $n$ members of $\{A_q: q \in \text{bdry } S\}$ meet in an $n$-dimensional set as well. Since $\bigcap \{A_q: q \in \text{bdry } S\} = \emptyset$, the number $f(n, 1) = n + 1$ is best possible.

We close with a stronger version of Theorem 1 which holds in the plane.

THEOREM 2. Let $S$ be a compact, connected, nonconvex set in $R^2$. Then for $k = 1$ or $k = 2$, $\dim \ker S \geq k$ if and only if every $g(k) = \max\{3, 6 - 2k\}$ inc points of $S$ are clearly visible from a common $k$-dimensional subset of $S$. The result is best possible.

PROOF. By [1, Lemma 4], if $S$ is a closed connected set in $R^2$, then $\ker S = \bigcap \{C_q \cap \text{conv } S: q \in \text{inc } S\}$. Using this result, it is not hard to show that Lemmas 2 and 3 above hold when $S$ is a compact, connected, nonconvex set in $R^2$ and when $\text{bdry } S$ is replaced by $\text{inc } S$. An easy adaptation of the proof of Theorem 1 completes the argument.

Examples 1 and 2 of [1] show that the result is best possible.
References