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A MONOTONICITY THEOREM FOR THE FAMILY $f_a(x) = a - x^2$

LEO JONKER

ABSTRACT. Let $f_a(x) = a - x^2$, $x \in [-\frac{1}{2} - \frac{1}{2}\sqrt{1+4a}, \frac{1}{2} + \frac{1}{2}\sqrt{1+4a}]$ and $a \in [0, 2]$. It is proved that if f_a has a periodic orbit of odd period n and if $b > a$, then f_b has a periodic orbit of period n. This is equivalent to the corresponding result for the function family $g_\lambda(x) = \lambda x(1-x)$, $x \in [0, 1]$, $\lambda \in [0, 4]$.

Among one-dimensional discrete dynamical systems those which are simplest and best understood are the unimodal functions of an interval into itself. These functions can be analysed with the help of Milnor and Thurston's kneading theory [**7**, **3**] which associates to the function f a formal power series $\nu(f) \in \mathbf{Z}[[t]]$ called variously the kneading invariant or (in [**7**]) the kneading determinant. The kneading invariant almost completely specifies the periodic orbits and completely specifies the topological entropy of f [**7**, **3**, **4**, **5**]. If in addition the function f has negative Schwarzian derivative, then the family of unimodal functions with the same kneading invariant either constitutes a conjugacy class of such functions or else is the union of two conjugacy classes [**2**]. Thus the kneading invariant almost characterizes f up to conjugacy. Milnor and Thurston showed that in a C^1 family of unimodal functions f_a the topological entropy is a continuous function of the parameter a (see [**7**]). The bifurcations in such a family are completely understood, and occur in a versal pattern that is related to Sarkovskii's ordering of the natural numbers [**6**]. However, there is no function family f_a for which it is known that $\nu(f_a)$ is a monotone function of a or that the topological entropy $h(f_a)$ is monotone in the parameter. Milnor and Thurston conjecture that the family of functions

$$g_\lambda(x) = \lambda x(1-x), \qquad x \in [0, 1],$$

has this property. Indeed computer calculations seem to indicate that $h(g_\lambda)$ is monotone increasing.

In this paper we present a modest result in this direction. It is known that the topological entropy increases monotonically with the appearance of periodic orbits of new periods. We prove the following.

THEOREM. *If g_λ has a periodic orbit of odd period n and if $\mu > \lambda$, then g_μ has a periodic orbit of period n.*

Let

$$f_a(x) = a - x^2$$

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where $x \in I_a = [-\frac{1}{2} - \frac{1}{2}\sqrt{1+4a}, \frac{1}{2} + \frac{1}{2}\sqrt{1+4a}\,]$ and where $a \in [0,2]$. Since $\varsigma(x) = (x/(1+\sqrt{1+4a}\,)) + \frac{1}{2}$ is a conjugacy between f_a and g_λ with $\lambda = (1+\sqrt{1+4a}\,)$, we prove the theorem for the function family f_a.

Both families are unimodal in the sense that $f'(x) > 0$ if $x < c$ and $f'(x) < 0$ if $x > c$ where c is the unique critical point.

Note that

$$\frac{\partial}{\partial a} f_a^n(x) = 1 - 2f_a^{n-1}(x) \cdot \frac{\partial}{\partial a} f_a^{n-1}(x).$$

Suppose now that $x_0, x_1, \ldots, x_{n-1}, \ldots$ is the orbit of a point x_0. Setting $\delta_i = \partial f_a^i(x_0)/\partial a$ we have $\delta_i = 1 - 2\delta_i x_i$, with $\delta_0 = 0$, $\delta_1 = 1$. If a is a boundary point of the set of a such that f_a has an orbit of period n, f_a is undergoing a "fold bifurcation" (see [**1**, **6**]). Then $\nu(f_a) = \beta(n)$ where

$$\beta(n) = \{1 - t - t^2 + (1-t)(t^3 + t^5 + \cdots + t^{n-2})\}/(1-t^n),$$

and close to 0 we have a periodic point $x_0 > 0$ of period n such that under iteration of f_a^n the interval $[0, x_0]$ converges to x_0.

Following Milnor and Thurston we define the kneading sequence of x as follows. Let

$$\epsilon_i(x) = \begin{cases} +1 & \text{if } f_a'(f_a^i(x_0)) > 0, \\ -1 & \text{if } f_a'(f_a^i(x_0)) < 0, \\ 0 & \text{if } f_a'(f_a^i(x_0)) = 0, \end{cases}$$

and let $\theta_i(x) = \epsilon_0(x)\epsilon_1(x)\ldots\epsilon_i(x)$. Finally let

$$\theta(x) = \sum_{i=0}^{\infty} \theta_i(x) t^i.$$

Then $\theta\colon I_a \to \mathbf{Z}[[t]]$ is a monotone decreasing function when we take lexicographic ordering on $\mathbf{Z}[[t]]$. In fact the coefficients of $\theta(x)$ are all ± 1 or 0, and if a zero occurs, all subsequent coefficients are zero. Furthermore, let $\tau\colon \mathbf{Z}[[t]] \to \mathbf{Z}[[t]]$ be the shift transformation

$$\tau \sum_{i=0}^{\infty} \theta_i t^i = \sum_{i=1}^{\infty} \theta_0 \theta_i t^{i-1}.$$

Then if $x \neq 0$ we have $\theta(f_a(x)) = \tau\theta(x)$. This and the fact that $\theta(x_0) = \pm\beta(n)$ (see [**5**]) allows us to determine the relative positions of the points $x_0, x_1, \ldots, x_{n-1}$ in the orbit of x_0.

First of all, $\tau^n(\beta(n)) = -\beta(n)$ whence $\theta(x_0) = -\beta(n)$. Straightforward calculation then reveals that the kneading coordinates of $x_0, \ldots, x_{n-1}$ are in the following lexicographic order

$$\begin{aligned} \theta(x_1) < \theta(x_{n-1}) < \theta(x_{n-3}) < \cdots < \theta(x_6) < \theta(x_4) < -\theta(x_2) \\ < \theta(x_3) < \theta(x_5) < \cdots < \theta(x_{n-4}) < \theta(x_{n-2}) < \theta(x_0) < 0. \end{aligned}$$

Since $\theta(-x_2) = -\theta(x_2)$ this implies that

$$0 < x_0 < x_{n-2} < \cdots < x_5 < x_3 < -x_2 < x_4 < x_6 < \cdots < x_{n-1} < x_1.$$

Now $|f_a(y) - f_a(x)| = |a - y^2 - a + x^2| = |y - x|\,|y + x|$. In other words, if f_a is monotone on the interval $[x, y]$ then its image is longer than $[x, y]$ by the factor

$|y+x|$. We apply this idea to two calculations.

$$\frac{x_1 - x_2}{x_1 - x_0} = x_1 + x_0.$$

But $x_1 - x_2 > x_1 + x_0$, and so $x_1 - x_0 > 1$. Thus $x_1 > 1$. Also,

$$\frac{x_4 - x_3}{-x_2 - x_3} = x_3 - x_2.$$

But $x_4 - x_3 > -x_2 - x_3$, and so $x_3 - x_2 > 1$. Since $-x_2 > x_3$ this implies $-x_2 > \frac{1}{2}$.

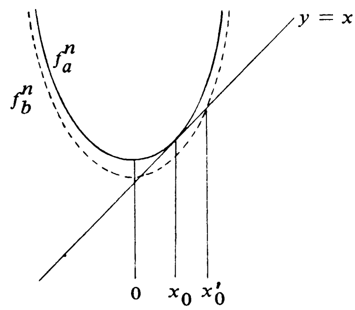
We now calculate δ_n.

$$\delta_2 = 1 - 2x_1\delta_1 = 1 - 2x_1 < -1; \quad \delta_3 = 1 - 2x_2\delta_2 < 0.$$

We continue by induction: If $\delta_k < 0$ for odd $k < n$, then because $x_k > 0$ we have $\delta_{k+1} = 1 - 2x_k \cdot \delta_k > 1$. But then because $x_{k+1} > -x_2 > \frac{1}{2}$, we have $\delta_{k+2} = 1 - 2x_{k+1} \cdot \delta_{k+1} < 0$. Thus,

$$\frac{\partial}{\partial a} f_a^n(x_0) = \delta_n < 0.$$

Because f_a is undergoing a fold bifurcation at x_0 and because $x_0 > 0$, the graph of f_a^n has the following form on a neighbourhood of $[0, x_0]$ (as opposed to the graph obtained by rotating 180°).



Since $\partial f_a^n(x_0)/\partial a < 0$, we see that for $b > a$ with $|b-a|$ sufficiently small the graph of f_b^n meets $y = x$ in a point x_0' of period n. This proves the theorem.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, K7L 3N6, CANADA