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A NOTE ON SPACES IN WHICH EVERY OPEN SET IS z-EMBEDDED

JOSE L. BLASCO[1]

ABSTRACT. Let Oz be the class of topological spaces in which every open set is z-embedded. In this note we prove the following: If Y is a dense subspace of the real line, then the spaces βY and $\beta Y - Y$ are not in Oz.

Introduction. A subset S of a topological space X is *z-embedded* in X if every zero-set in S is the intersection of S with a zero-set in X. (A *zero-set* is the set of zeros of a real-valued continuous function.) Blair [**1**] studied the class Oz of topological spaces in which every open set is z-embedded. This class includes all perfectly normal spaces, all extremally disconnected spaces and all products of separable metric spaces. For basic results of the class Oz see [**1** and **2**].

Blair [**1**] asked if the spaces βR, βQ and $\beta Q - Q$ are in Oz. In [**6**] Terada characterizes a class of spaces whose Stone-Čech compactifications are in Oz. As an application of his characterizations he showed that both βR and βQ do not belong to Oz. E. K. van Douwen [**4**] has proved that $\beta Q - Q$ does not belong to Oz.

In this note we shall prove that for Y dense in R, the spaces βY and $\beta Y - Y$ are not in Oz.

Preliminaries. Throughout this paper we adopt the notation and terminology of [**5**]. βX and υX denote respectively the Stone-Čech compactification and the Hewitt realcompactification of the Tychonoff space X. $Z(X)$ denotes the family of all zero-sets in X. The remainder $\beta X - X$ is always denoted by X^*. R is the space of all real numbers with the usual topology, Z is the subspace of all integer numbers and N is the subspace of all positive integers.

Let S be a subset of the topological space X. The G_δ-closure of S is the set $G_\delta\text{-cl}_X S$ of all points $p \in X$ satisfying the condition that whenever G is a G_δ-set containing p, then $G \cap S \neq \emptyset$. For Tychonoff X, $G_\delta\text{-cl}_X S$ is precisely all $p \in X$ for which each zero-set about p meets S. The following fact is needed: (a) [**3**, 1.1(b)] If S is z-embedded in the Tychonoff space X, then the G_δ-closure of S in υX is υS. The set S is said to be G_δ-dense in X if $X = G_\delta\text{-cl}_X S$.

The result. In the sequel, Y will be a dense subspace of R. Let $S = \{a(n): n \in Z\}$ be a copy of Z contained in Y such that $a(n+1) - a(n) \geq 1$ for $n \in Z$. Consider the following closed subsets of Y, $I = \bigcup\{[a(2n), a(2n+1)] \cap Y: n \in Z\}$ and $J = \bigcup\{[a(2n-1), a(2n)] \cap Y: n \in Z\}$. Since Y is a metric space, I and J are zero-sets in Y. Therefore $\beta Y = \text{cl}_{\beta Y} I \cup \text{cl}_{\beta Y} J$ and $\text{cl}_{\beta Y} S = \text{cl}_{\beta Y} I \cap \text{cl}_{\beta Y} J$. We

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need the following fact:

(b) $\quad \mathrm{cl}_{\beta Y} S - S \subset \mathrm{cl}_{Y^*}((\mathrm{cl}_{\beta Y} I - \mathrm{cl}_{\beta Y} S) \cap Y^*) \cap \mathrm{cl}_{Y^*}((\mathrm{cl}_{\beta Y} J - \mathrm{cl}_{\beta Y} S) \cap Y^*)$.

Indeed, let p be a point in $\mathrm{cl}_{\beta Y} S - S$ and let V be a closed neighborhood of p in Y^*. There exists an open set W in βY such that $p \in W$ and $Y^* \cap \mathrm{cl}_{\beta Y} W \subset V$. Since the set $W \cap S$ is infinite, we can choose a closed (in Y) copy E of N such that $E \subset I \cap W$ and $E \cap S = \emptyset$. Then

$$\emptyset \neq Y^* \cap \mathrm{cl}_{\beta Y} E \subset Y^* \cap \mathrm{cl}_{\beta Y} I \cap \mathrm{cl}_{\beta Y} W \subset (\mathrm{cl}_{\beta Y} I) \cap V.$$

Since E and S are disjoint zero-sets in Y it follows that $\mathrm{cl}_{\beta Y} E \cap \mathrm{cl}_{\beta Y} S = \emptyset$ and therefore the set $(\mathrm{cl}_{\beta Y} I - \mathrm{cl}_{\beta Y} S) \cap V$ is nonempty. Hence

$$p \in \mathrm{cl}_{Y^*}((\mathrm{cl}_{\beta Y} I - \mathrm{cl}_{\beta Y} S) \cap Y^*).$$

We can replace I by J in the above argument. The inclusion is now proved.

Let $X = \beta Y - \mathrm{cl}_{\beta Y} S$.

ASSERTION 1. *X is not C^*-embedded in $\beta Y - S$.*

PROOF. The family $\{\mathrm{cl}_{\beta Y} I - \mathrm{cl}_{\beta Y} S, \mathrm{cl}_{\beta Y} J - \mathrm{cl}_{\beta Y} S\}$ is a partition of X, so the characteristic function (in X) f of the set $\mathrm{cl}_{\beta Y} I - \mathrm{cl}_{\beta Y} S$ is continuous on X. According to (b), f has no continuous extension to $\beta Y - S$, therefore X is not C^*-embedded in $\beta Y - S$.

ASSERTION 2. *The G_δ-closure in βY of $Y^* \cap X$ is Y^*.*

PROOF. Since the points of Y are zero-sets in βY, it follows that

$$G_\delta\text{-}\mathrm{cl}_{\beta Y}(Y^* \cap X) \subset Y^*.$$

Suppose now that $p \in Y^*$ is not in G_δ-$\mathrm{cl}_{\beta Y}(Y^* \cap X)$. Then there exists a zero-set T in βY such that $p \in T \subset \beta Y - (Y^* \cap X)$. Moreover, since Y is realcompact [**5**, Corollary 8.15] there is a zero-set F in βY such that $p \in F \subset Y^*$. Let h be a real-valued continuous function on βY such that $h^{-1}(\{0\}) = T \cap F \subset Y^* \cap \mathrm{cl}_{\beta Y} S$. The reciprocal g of $h|X \cup S$ is continuous and unbounded on $X \cup S$, consequently g must be unbounded on some countable closed subspace H of Y which misses S. Since H and S are disjoint zero-sets in Y we have that $\mathrm{cl}_{\beta Y} H \cap \mathrm{cl}_{\beta Y} S = \emptyset$, therefore g must be unbounded on $\mathrm{cl}_{\beta Y} H \subset X \cup S$, which is a contradiction. This shows $Y^* = G_\delta$-$\mathrm{cl}_{\beta Y}(Y^* \cap X)$.

ASSERTION 3. *The space βY does not belong to Oz.*

PROOF. Suppose that $\beta Y \in Oz$. Then X is z-embedded in βY and according to (a), $\upsilon X = G_\delta$-$\mathrm{cl}_{\beta Y} X$. Since the points of Y are zero-sets in βY, it follows that $\upsilon X \subset \beta Y - S$. From Assertion 2 we have $\upsilon X = \beta Y - S$, which contradicts Assertion 1. Hence $\beta Y \notin Oz$.

A subset S of a space E is a *generalized cozero-set* in case for every neighborhood V of S there is a cozero-set P such that $S \subset P \subset V$. It is known that every generalized cozero-set in a normal space is normal and z-embedded [**1**, Theorem 2.5].

ASSERTION 4. *Y^* is realcompact and z-embedded in βY.*

PROOF. Since every compact subset of Y is a zero-set in βY, we have that Y^* is a generalized cozero-set in βY. According to [**1**, Theorem 2.5], Y^* is z-embedded in βY. On the other hand, since every cozero-set in βY is realcompact and $Y^* = \bigcap\{\beta Y - \{p\}: p \in Y\}$, it follows that Y^* is realcompact.

ASSERTION 5. *Y^* does not belong to Oz.*

PROOF. From Assertion 4, Y^* is z-embedded in βY. Thus by Assertion 2, $G_\delta\text{-cl}_{Y^*}(Y^* \cap X) = Y^*$. Hence, if $Y^* \in Oz$ we have that $Y^* \cap X$ is G_δ-dense and z-embedded in Y^* [**1**, Theorem 5.1]. Therefore $Y^* = \upsilon(Y^* \cap X)$ and $Y^* \cap X$ is C-embedded in Y^*.

On the other hand, the set $(\text{cl}_{\beta Y} I - \text{cl}_{\beta Y} S) \cap Y^*$ is clopen in $Y^* \cap X$, therefore its characteristic function (in $Y^* \cap X$) is continuous. According to (b), this function has no continuous extension to Y^*. This contradiction shows that $Y^* \notin Oz$.

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CATEDRA DE MATEMATICAS II, FACULTAD DE CIENCIAS, BURJASOT, VALENCIA, SPAIN