A NOTE ON SPACES IN WHICH EVERY OPEN SET IS $z$-EMBEDDED

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ABSTRACT. Let $Oz$ be the class of topological spaces in which every open set is $z$-embedded. In this note we prove the following: If $Y$ is a dense subspace of the real line, then the spaces $\beta Y$ and $\beta Y - Y$ are not in $Oz$.

Introduction. A subset $S$ of a topological space $X$ is $z$-embedded in $X$ if every zero-set in $S$ is the intersection of $S$ with a zero-set in $X$. (A zero-set is the set of zeros of a real-valued continuous function.) Blair [1] studied the class $Oz$ of topological spaces in which every open set is $z$-embedded. This class includes all perfectly normal spaces, all extremally disconnected spaces and all products of separable metric spaces. For basic results of the class $Oz$ see [1 and 2].

Blair [1] asked if the spaces $\beta R$, $\beta Q$ and $\beta Q - Q$ are in $Oz$. In [6] Terada characterizes a class of spaces whose Stone-Čech compactifications are in $Oz$. As an application of his characterizations he showed that both $\beta R$ and $\beta Q$ do not belong to $Oz$. E. K. van Douwen [4] has proved that $\beta Q - Q$ does not belong to $Oz$.

In this note we shall prove that for $Y$ dense in $R$, the spaces $\beta Y$ and $\beta Y - Y$ are not in $Oz$.

Preliminaries. Throughout this paper we adopt the notation and terminology of [5]. $\beta X$ and $vX$ denote respectively the Stone-Čech compactification and the Hewitt realcompactification of the Tychonoff space $X$. $Z(X)$ denotes the family of all zero-sets in $X$. The remainder $\beta X - X$ is always denoted by $X^*$. $R$ is the space of all real numbers with the usual topology, $Z$ is the subspace of all integer numbers and $N$ is the subspace of all positive integers.

Let $S$ be a subset of the topological space $X$. The $G_\delta$-closure of $S$ is the set $G_\delta$-cl$_X S$ of all points $p \in X$ satisfying the condition that whenever $G$ is a $G_\delta$-set containing $p$, then $G \cap S \neq \emptyset$. For Tychonoff $X$, $G_\delta$-cl$_X S$ is precisely all $p \in X$ for which each zero-set about $p$ meets $S$. The following fact is needed: (a) [3, 1.1(b)] If $S$ is $z$-embedded in the Tychonoff space $X$, then the $G_\delta$-closure of $S$ in $\beta X$ is $\nu S$. The set $S$ is said to be $G_\delta$-dense in $X$ if $X = G_\delta$-cl$_X S$.

The result. In the sequel, $Y$ will be a dense subspace of $R$. Let $S = \{a(n): n \in Z\}$ be a copy of $Z$ contained in $Y$ such that $a(n + 1) - a(n) \geq 1$ for $n \in Z$. Consider the following closed subsets of $Y$, $I = \bigcup \{[a(2n), a(2n + 1)] \cap Y: n \in Z\}$ and $J = \bigcup \{[a(2n - 1), a(2n)] \cap Y: n \in Z\}$. Since $Y$ is a metric space, $I$ and $J$ are zero-sets in $Y$. Therefore $\beta Y = cl_{\beta Y} I \cup cl_{\beta Y} J$ and $cl_{\beta Y} S = cl_{\beta Y} I \cap cl_{\beta Y} J$. We
need the following fact:

(b) \( \text{cl}_{\beta Y} S - S \subseteq \text{cl}_{\gamma^*}((\text{cl}_{\beta Y} I - \text{cl}_{\beta Y} S) \cap \gamma^*) \cap \text{cl}_{\gamma^*}((\text{cl}_{\beta Y} J - \text{cl}_{\beta Y} S) \cap \gamma^*). \)

Indeed, let \( p \) be a point in \( \text{cl}_{\beta Y} S - S \) and let \( V \) be a closed neighborhood of \( p \) in \( \gamma^* \). There exists an open set \( W \) in \( \beta Y \) such that \( p \in W \) and \( \gamma^* \cap \text{cl}_{\beta Y} W \subseteq V \). Since the set \( W \cap S \) is infinite, we can choose a closed (in \( \beta Y \)) copy \( E \) of \( N \) such that \( E \subseteq I \cap W \) and \( E \cap S = \emptyset \). Then

\[ \emptyset \neq \gamma^* \cap \text{cl}_{\beta Y} E \subseteq \gamma^* \cap \text{cl}_{\beta Y} I \cap \text{cl}_{\beta Y} W \subseteq (\text{cl}_{\beta Y} I) \cap V. \]

Since \( E \) and \( S \) are disjoint zero-sets in \( \gamma \) it follows that \( \text{cl}_{\beta Y} E \cap \text{cl}_{\beta Y} S = \emptyset \) and therefore the set \( (\text{cl}_{\beta Y} I - \text{cl}_{\beta Y} S) \cap V \) is nonempty. Hence

\[ p \in \text{cl}_{\gamma^*}((\text{cl}_{\beta Y} I - \text{cl}_{\beta Y} S) \cap \gamma^*). \]

We can replace \( I \) by \( J \) in the above argument. The inclusion is now proved.

Let \( X = \beta Y - \text{cl}_{\beta Y} S \).

**Assertion 1.** \( X \) is not \( C^* \)-embedded in \( \beta Y - S \).

**Proof.** The family \( \{ \text{cl}_{\beta Y} I - \text{cl}_{\beta Y} S, \text{cl}_{\beta Y} J - \text{cl}_{\beta Y} S \} \) is a partition of \( X \), so the characteristic function (in \( X \)) \( f \) of the set \( \text{cl}_{\beta Y} I - \text{cl}_{\beta Y} S \) is continuous on \( X \). According to (b), \( f \) has no continuous extension to \( \beta Y - S \), therefore \( X \) is not \( C^* \)-embedded in \( \beta Y - S \).

**Assertion 2.** The \( G_\delta \)-closure in \( \beta Y \) of \( \gamma^* \cap X \) is \( \gamma^* \).

**Proof.** Since the points of \( \gamma \) are zero-sets in \( \beta Y \), it follows that

\[ G_{\delta^*}\text{cl}_{\beta Y}(\gamma^* \cap X) \subseteq \gamma^*. \]

Suppose now that \( p \in \gamma^* \) is not in \( G_{\delta^*}\text{cl}_{\beta Y}(\gamma^* \cap X) \). Then there exists a zero-set \( T \) in \( \beta Y \) such that \( p \in T \subseteq \beta Y - (\gamma^* \cap X) \). Moreover, since \( \gamma \) is realcompact [5, Corollary 8.15] there is a zero-set \( F \) in \( \beta Y \) such that \( p \in F \subseteq Y^* \). Let \( h \) be a real-valued continuous function on \( \beta Y \) such that \( h^{-1}(\{0\}) = T \cap F \subseteq \gamma^* \cap \text{cl}_{\beta Y} S \). The reciprocal \( g \) of \( h|X \cup S \) is continuous and unbounded on \( X \cup S \), consequently \( g \) must be unbounded on some countable closed subspace \( H \) of \( \gamma \) which misses \( S \). Since \( H \) and \( S \) are disjoint zero-sets in \( \gamma \) we have that \( \text{cl}_{\beta Y} H \cap \text{cl}_{\beta Y} S = \emptyset \), therefore \( g \) must be unbounded on \( \text{cl}_{\beta Y} H \subseteq X \cup S \), which is a contradiction. This shows \( \gamma^* = G_{\delta^*}\text{cl}_{\beta Y}(\gamma^* \cap X) \).

**Assertion 3.** The space \( \beta Y \) does not belong to \( Oz \).

**Proof.** Suppose that \( \beta Y \in Oz \). Then \( X \) is \( z \)-embedded in \( \beta Y \) and according to (a), \( \nu X = G_{\delta^*}\text{cl}_{\beta Y} X \). Since the points of \( Y \) are zero-sets in \( \beta Y \), it follows that \( \nu X \subseteq \beta Y - S \). From Assertion 2 we have \( \nu X = \beta Y - S \), which contradicts Assertion 1. Hence \( \beta Y \notin Oz \).

A subset \( S \) of a space \( E \) is a generalized cozero-set in case for every neighborhood \( V \) of \( S \) there is a cozero-set \( P \) such that \( S \subseteq P \subseteq V \). It is known that every generalized cozero-set in a normal space is normal and \( z \)-embedded [1, Theorem 2.5].

**Assertion 4.** \( \gamma^* \) is realcompact and \( z \)-embedded in \( \beta Y \).
PROOF. Since every compact subset of $Y$ is a zero-set in $\beta Y$, we have that $Y^*$ is a generalized cozero-set in $\beta Y$. According to [1, Theorem 2.5], $Y^*$ is $z$-embedded in $\beta Y$. On the other hand, since every cozero-set in $\beta Y$ is realcompact and $Y^* = \bigcap\{\beta Y - \{p\}: p \in Y\}$, it follows that $Y^*$ is realcompact.

ASSERTION 5. $Y^*$ does not belong to $Oz$.

PROOF. From Assertion 4, $Y^*$ is $z$-embedded in $\beta Y$. Thus by Assertion 2, $G_\delta$-cl$_\gamma(Y^* \cap X) = Y^*$. Hence, if $Y^* \in Oz$ we have that $Y^* \cap X$ is $G_\delta$-dense and $z$-embedded in $Y^*$ [1, Theorem 5.1]. Therefore $Y^* = v(Y^* \cap X)$ and $Y^* \cap X$ is $C$-embedded in $Y^*$.

On the other hand, the set $(cl_{\beta Y} I - cl_{\beta Y} S) \cap Y^*$ is clopen in $Y^* \cap X$, therefore its characteristic function (in $Y^* \cap X$) is continuous. According to (b), this function has no continuous extension to $Y^*$. This contradiction shows that $Y^* \notin Oz$.

REFERENCES

1. R. L. Blair, Spaces in which special sets are $z$-embedded, Canad. J. Math. 28 (1976), 673–690.

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