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A CELLULAR MAP BETWEEN NONHOMEOMORPHIC POLYHEDRA WHOSE NONDEGENERACY SET IS A NULL SEQUENCE

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ABSTRACT. A cellular map between nonhomeomorphic polyhedra whose nondegeneracy set consists of a null sequence of cellular sets is constructed. The construction begins with Daverman's example of a cellular map between nonhomeomorphic polyhedra and uses an amalgamation technique.

Early results obtained concerning cellular maps between polyhedra [**6, 7**] seemed to indicate that such maps might indeed behave as nicely as cellular maps between n-manifolds, $n \neq 4$, namely, such maps are approximable by homeomorphisms [**1, 5, 10**]. However, Daverman recently constructed an example of a cellular map between nonhomeomorphic polyhedra. (See [**8**].) The knowledge that cellular maps between polyhedra need not always be approximable by homeomorphisms leads one to try to find a "simplest" such example. Since any cellular map having only a finite number of nondegenerate point preimages is approximable by homeomorphisms, the next case to consider would be a map with a countable number of nondegenerate point preimages, each being a cellular subset of the domain. In light of the Bing shrinking criterion, one might be led to expect that if this sequence of cellular sets was in fact a null sequence, then the cellular map would be approximable by homeomorphisms. However, examples of Bing [**3**], Daverman [**4**], and Wright [**12**] of nonshrinkable decompositions of manifolds with a null sequence of cellular sets as the nondegeneracy set indicate that more is involved. In fact, one can modify Daverman's cellular map between nonhomeomorphic polyhedra to obtain a cellular map between polyhedra, with nondegeneracy set consisting of a null sequence of cellular sets, which cannot be approximated by homeomorphisms. The purpose of this paper is to construct that map.

1. Preliminaries. A *polyhedron* P is a subset of some Euclidean space $\mathbf{R}^n$ such that each point $b \in P$ has a neighborhood $N = bL$, the join of b and a compact subset L of P [**9**]. A homotopy $h_t\colon P \to P$, $0 \leq t \leq 1$, for which h_t is a homeomorphism when $t < 1$ is a *pseudoisotopy*. A compact subset X of P is *cellular* in P if there is a pseudoisotopy $h_t\colon P \to P$ such that X is the only nondegenerate point preimage of h_1. A proper surjection $f\colon P \to Q$ between polyhedra is a *cellular map* if for each $y \in Q$, $f^{-1}(y)$ is a cellular subset of P. The *nondegeneracy set* of f is $N_f = \{f^{-1}(y) | y \in Q \text{ and } f^{-1}(y) \text{ is not a point}\}$.

The *intrinsic dimension* of a point x in P, denoted $I(x, P)$, is given by $I(x, P) = \max\{n \in \mathbf{Z} |$ there is an open embedding $h\colon R^n \times \mathrm{cL} \to P$ with L a compact polyhedron and $h(\mathbf{R}^n \times \mathrm{cL})$ a neighborhood of $h(0 \times c) = x\}$, where cL is the open

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cone on L [**2**]. A *cellular neighborhood* in P is an open subset of P homeomorphic to $\mathbf{R}^n \times \mathrm{cL}$, again with L a compact polyhedron and cL the open cone on L. The *intrinsic n-skeleton* of P is $P^{(n)} = \{x \in P | I(x,P) \leq n\}$, and the *intrinsic n-stratum* of P is $P[n] = P^{(n)} - P^{(n-1)}$. Note that $P[n]$ is a topological n-manifold.

One theorem from [**6**] will be stated here for later reference.

THEOREM 1. *The following are equivalent:*

(1) *X is cellular in P.*

(2) *The projection π: $P \to P/X$ is approximable by homeomorphisms.*

(3) *$X = \bigcap_{i=1}^{\infty} W_i$, where the W_i's are homeomorphic cellular neighborhoods with $\overline{W_{i+1}} \subset W_i$.*

2. Daverman's example. Let W be a contractible $(n+1)$-manifold, $n \geq 3$, whose boundary is a nonsimply connected homology n-sphere H, and choose $w \in \operatorname{int} W$. The polyhedron P is given by

$$P = (W \times S^1) \underset{w \times S^1}{\cup} (W \times S^1),$$

two copies of $W \times S^1$ identified along $w \times S^1$ by the identity map. Let $\tilde{W}$ be a submanifold of W such that $\tilde{W}$ is homeomorphic to W, $w \in \tilde{W}$, and $W - \operatorname{int} \tilde{W} \simeq H \times [0,1]$.

The polyhedron Q is defined by

$$Q = (\mathrm{cH} \times S^1) \underset{c \times S^1}{\cup} (\mathrm{cH} \times S^1),$$

with cH being the standard cone on the homology sphere H. The cellular map f: $P \to Q$ then takes each $(W \times y) \cup_{w \times y} (W \times y)$ to $(\mathrm{cH} \times y) \cap_{c \times y} (\mathrm{cH} \times y)$ by sending the cellular subset $(\tilde{W} \times y) \cup_{w \times y} (\tilde{W} \times y)$ to the point $c \times y$. (See [**7**] for a more detailed discussion of the example.)

3. Modifying cellular maps. In order to construct the desired example, it will be necessary to modify the map f: $P \to Q$ by shrinking out certain nondegenerate point preimages. The following proposition and Theorem 1 are needed to conclude that the resulting map is cellular. The statement of the proposition is a slight variation on that of Proposition 1.3 of Walsh [**11**], and the proof included here is essentially his.

PROPOSITION 2. *Let $\tilde{f}$: $\tilde{P} \to \tilde{Q}$ be a cellular map between compact polyhedra. If $\tilde{h}_t$: $P \to P$, $0 \leq t < 1$, is an isotopy such that $\tilde{g} = \lim_{t \to 1} \tilde{f} \circ \tilde{h}_t$: $\tilde{P} \to \tilde{Q}$ is a continuous function, then $\tilde{g}$ is also cellular.*

PROOF. Choose $y \in \tilde{Q}$. Since $\tilde{f}^{-1}(y)$ is cellular in $\tilde{P}$, it follows from Theorem 1 that there is a sequence of homeomorphic cellular neighborhoods $\{W_i\}_{i \in \mathbf{N}}$ such that $\overline{W}_{i+1} \subset W_i$ and $\tilde{f}^{-1}(y) = \bigcap_{i=1}^{\infty} W_i$. Given $\epsilon > 0$, it is sufficient to show that there is some t, $0 \leq t < 1$, and integer k such that

$$\tilde{g}^{-1}(y) \subset \tilde{h}_t^{-1}(W_k) \subset N_\epsilon(g^{-1}(y)).$$

Let $U \subseteq \tilde{g}(N_\epsilon(\tilde{g}^{-1}(y)))$ be an open neighborhood of y with $\tilde{g}^{-1}(U) \subseteq N_\epsilon(\tilde{g}^{-1}(y))$. There is a $t' < 1$ such that for $t \geq t'$, $(\tilde{f} \circ \tilde{h}_t)(y) \subseteq \tilde{g}^{-1}(U)$. Choose k so that $\tilde{f}(W_k) \subseteq U$ and $(\tilde{f} \circ \tilde{h}_t)^{-1}(\tilde{f}(W_k)) \subseteq \tilde{g}^{-1}(U)$ for $t \geq t'$. Let $U' \subseteq \tilde{f}(W_k)$ be an open neighborhood of y with $\tilde{f}^{-1}(U') \subseteq W_k$. There is an s, $1 > s \geq t'$, such that for $t > s$, $\tilde{g}^{-1}(y) \subseteq (\tilde{f} \circ \tilde{h}_t)^{-1}(U')$. Now for $1 > t > s$,

$$\begin{aligned}\tilde{g}^{-1}(y) &\subseteq (\tilde{f} \circ \tilde{h}_t)^{-1}(U') = \tilde{h}_t^{-1}(\tilde{f}^{-1}(U')) \subseteq \tilde{h}_t^{-1}(W_k) \\ &\subseteq \tilde{h}_t^{-1}(\tilde{f}^{-1}(\tilde{f}(W_k))) = (\tilde{f} \circ \tilde{h}_t)^{-1}(\tilde{f}(W_k)) \subseteq \tilde{g}^{-1}(U) \subseteq N_\epsilon(\tilde{g}^{-1}(y)).\end{aligned}$$

Thus we have $\tilde{g}^{-1}(y) \subseteq \tilde{h}_t^{-1}(W_k) \subseteq N_\epsilon(g^{-1}(y))$ for some $t < 1$.

The example. Let $f\colon P \to Q$ be Daverman's example described earlier, and let $D = \{y_i\}$ be a countable dense subset of $c \times S^1 = Q[1]$. Since each $f^{-1}(y_i)$ is cellular in P, there exist cellular neighborhoods $W_{i,k}$ such that $f^{-1}(y_i) = \bigcap_{k=1}^{\infty} W_{i,k}$. Choose $k(1)$ so that $\operatorname{diam} f(W_{1,k(1)}) < 1$. It follows from Proposition 1.5 of [**6**] that there is a homeomorphism $H_1\colon P \to P$ such that

(1) H_1 is isotopic to the identity map on P with compact support in $W_{1,k(1)} - P[1]$.

(2) $\operatorname{diam} H_1(f^{-1}(y_1)) < 1$.

Inductively now, choose an integer $k(i) > k(i-1)$ such that $f(W_{r,k(i)}) \cap f(W_{s,k(i)}) = \emptyset$ for $1 \leq r, s \leq i$ and $r \neq s$, and $\operatorname{diam} f(W_{r,k(i)}) < 1/2^i$. As above, there is a homeomorphism $H_i\colon P \to P$ such that

(1) H_i is isotopic to H_{i-1} with compact support in $H_{i-1}(\bigcup_{r \leq i} W_{r,k(i)}) - P[1]$,

(2) $\operatorname{diam} H_i(f^{-1}(y_r)) < 1/i$ for $1 \leq r \leq i$.

Now let $h_t\colon P \to P$ be the isotopy obtained by piecing together the above isotopies. That is, by defining $h_{i/(i+1)} = H_i$, and letting $h_t\colon P \to P$, $i/(i+1) \leq t \leq (i+1)/(i+2)$, be the isotopy between H_i and H_{i+1}. Then $\lim_{t \to 1} f \circ h_t^{-1} = \tilde{f}$ will be a continuous function from P onto Q which is cellular by Proposition 2. Note that $\tilde{f}$ is 1-1 over D. Also, we apply Theorem 4.2 of [**6**] to approximate the restriction of $\tilde{f}$ to $P - \tilde{f}^{-1}(Q[1])$ by a homeomorphism which extends to $\tilde{f}$ on $\tilde{f}^{-1}(Q[1])$. Thus it may be assumed that $\tilde{f}$ is 1-1 over $(Q - Q[1]) \cup D$.

At this point, the image of the nondegeneracy set of $\tilde{f}$ is 0-dimensional. The amalgamation idea used by Edwards [**5**], Daverman [**4**], and Wright [**12**] is now employed. Let $N_i = \{\tilde{f}^{-1}(y) \mid \operatorname{diam} \tilde{f}^{-1}(y) \geq 1/i\}$. Each N_i is a closed subset of P, and hence $\tilde{f}(N_i)$ is a closed subset of $Q[1]$. Cover $\tilde{f}(N_i)$ with a finite collection of disjoint arcs $A, \ldots, A_{k(1)}$, each having endpoints in D. Inductively, $\tilde{f}(N_{i+1}) - \tilde{f}(N_i)$ is a closed subset of $Q[1] - \bigcup_{j < k(i)} A_j$. Cover $\tilde{f}(N_{i+1}) - \tilde{f}(N_i)$ with a finite collection of arcs $A_{k(i)+1}, \ldots, A_{k(i+1)}$, each having diameter less than $1/(i+1)$ and lying in $Q[1] - \bigcup_{j \leq k(i)} A_j$.

Consider the collection $\{A_i\}_{i \in \mathbf{N}}$ of arcs in $Q[1]$. It is a null collection of arcs, and by the choice of the N_i's, $\{\tilde{f}^{-1}(A_i)\}_{i \in \mathbf{N}}$ will also be a null sequence. It remains to show that each $\tilde{f}^{-1}(A_i)$ is cellular in P and that the decomposition G of P whose nondegenerate elements are $\{\tilde{f}^{-1}(A_i)\}_{i \in \mathbf{N}}$ yields Q. Then $\pi\colon P \to P/\{\tilde{f}^{-1}(A_i)\}_{i \in \mathbf{N}} \simeq Q$ will be the desired map.

First note that $Q/\{A_i\}_{i \in \mathbf{N}} \simeq Q$. Identifying $c \times S^1$ with S^1, there is a pseudoisotopy $g_t\colon S^1 \to S^1$ such that the degenerate point preimages of g_1 are $\{A_i\}_{i \in \mathbf{N}}$. Letting $\mathrm{c}\mathrm{H}^n = (H^n \times [0,2])/(H^n \times \{0\})$, define $G_r\colon \mathrm{c}\mathrm{H}^n \times S^1 \to \mathrm{c}\mathrm{H}^n \times S^1$ by

$$G_r(x,t,s) = \begin{cases} (x,t,g_{r(1-t)}(s)), & 0 \leq t \leq 1, \\ (x,t,s), & t \geq 1, \end{cases}$$

where $r \in [0,1]$, $x \in H^n$, $t \in [0,2]$, and $s \in S^1$. Now G_1: $Q \to Q$ has $\{A_i\}_{i \in \mathbf{N}}$ as its nondegeneracy set, and it follows that $Q/\{A_i\}_{i \in \mathbf{N}} \simeq Q$.

Also, the composition $G_1 \circ f$: $P \to Q$ is a cellular map where nondegenerate point preimages are $\{B_i \times \tilde{W}\}_{i \in \mathbf{N}}$, where B_i is the arc in $P[1]$ which is mapped homeomorphically onto A_i by f. If we consider the map $f^* = \lim_{t \to 1} G_1 \circ f \circ h_t^{-1} = G_1 \circ \tilde{f}$, then again by Proposition 2, f^* is a cellular map. But if $y_i = G_1(A_i)$, then

$$(f^*)^{-1}(y_i) = \lim_{t \to 1} h_t(f^{-1} \circ G_1^{-1}(y_i)) = \lim_{t \to 1} h_t(f^{-1}(A_i)) = \tilde{f}^{-1}(A_i).$$

Thus each $\tilde{f}^{-1}(A_i)$ is cellular in P. Now π: $P \to P/\{\tilde{f}^{-1}(A_i)\}_{i \in \mathbf{N}}$ is a cellular map and $P/\{\tilde{f}^{-1}(A_i)\}_{i \in \mathbf{N}} \simeq \tilde{f}(P)/\{A_i\}_{i \in \mathbf{N}} \simeq Q/\{A_i\}_{i \in \mathbf{N}} \simeq Q$. Thus π: $P \to Q$ is the desired map.

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