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INVOLUTIONS ON DOLD MANIFOLDS

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ABSTRACT. Using the techniques of Bredon, some results are obtained concerning the possible cohomology of the fixed set of a smooth involution on a Dold manifold.

In this paper we are concerned with the possible $Z/2$ cohomologies of components of the fixed set F of a smooth involution T on a Dold manifold $P(m,n) = S^m \times_{Z/2} CP(n)$ which satisfies the condition that the fibration

$$\pi\colon P(m,n) \times_{T\times(-1)} S^\infty \to RP(\infty)$$

is totally nonhomologous to zero [**6**, **8**]. As a general reference, see the last chapter of [**2**].

We know [8] that

$$H^*(P(m,n)) = Z/2[c,d]/(c^{m+1}, d^{n+1})$$

where degree $d = 2$, degree $c = 1$. For any component of F there are lifts γ and δ of c and d to $H^*(P(m,n) \times_{Z/2} S^\infty)$ which restrict to zero in $H^*(RP(\infty))$. The pullbacks of γ and δ to $H^*(F \times BZ/2)$ will be denoted by $ut + v$ and $xt^2 + yt + z$, respectively. The restrictions of u, v, x, y, z to any component F_i of F will be denoted u_i, v_i, x_i, y_i, z_i.

THEOREM 1. *For any component F_1 of F, $H^*(F_1)$ is generated by:*
(i) *a single class of degree less than or equal to* 2; *or*
(ii) *two linearly independent classes of degree* 1; *or*
(iii) *two classes $a \in H^1(F_1)$, $b \in H^2(F_1)$, with b not a multiple of a^2 and with* $\mathrm{Sq}^1 b = ab$.

PROOF. Pick γ and δ using F_1. By [**1**, **3**], v_1, y_1, z_1 generate $H^*(F_1)$, but by [**4**] two of these suffice.

Suppose $H^*(F_1)$ is generated by classes $a \in H^1$ and $b \in H^2$ with b not a multiple of a^2. If v_1 were zero, there would be $P_j \in \{0,1\}$ such that, for some large integer p,

$$t^p y_1 = P_j(y_1 t + z_1)^j t^{p+1-2j},$$

so that $z_1 = p_2 y_1^2$. Thus $v_1 \neq 0$ and so z_1 and v_1 will generate $H^*(F_1)$. Since v_1 and z_1 are the restrictions of c and d, $\mathrm{Sq}^1 z_1 = v_1 z_1$. $\square$

REMARK. The condition that π be totally nonhomologous to zero cannot be dropped. Let T be the involution on $P(m, 2j+1)$ defined on coordinates by

$$T((x_0,\dots,x_m),[z_0,\dots,z_{2j+1}]) = ((-x_0,x_1,\dots,x_m),[-z_1,z_0,\dots,-z_{2j+1},z_{2j}]).$$

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The fixed set of T is $S^{m-1} \times CP(j)$ which, for $m > 2$ and $j > 1$, does not have the cohomology described by Theorem 1.

THEOREM 2. *The fixed set of an involution of the given type on $P(m,n)$ has at most three components.*

PROOF. Pick γ and δ using any component. There are $\chi_i \in \{0,1\}$ such that

$$\mathrm{Sq}^1\delta = \chi_1\gamma t^2 + \chi_2\delta t + \chi_3\gamma^2 t + \gamma\delta. \tag{1}$$

Pulling this back to $F \times BZ/2$ and equating coefficients of t^3 yields $ux = (\chi_1 + \chi_3)u + \chi_2 x$. Thus $\{1, u, x\}$ generates $H^0(F)$. $\square$

REMARK. If, for some choice of γ, $\gamma^{m+1} = 0$, then u is also zero and F can have at most two components.

For the remainder of the paper, we will consider only the cases where $m = 1$ and $n \geq 1$. We write $M \sim N$ to mean that $H^*(M; Z/2) \cong H^*(N; Z/2)$.

THEOREM 3. *Let T be an involution of the given type on $P(1,n)$ with nonempty fixed set F. Then*

(i) *$F \sim RP(1) \times RP(n)$; or*
(ii) *$F \sim RP((n-1)\epsilon \bigoplus \sigma)$ where ϵ and σ are, respectively, the trivial and nontrivial real line bundles over $RP(1)$; or*
(iii) *$F \sim P(1,r) + P(1, n-r-1)$, $0 \leq r \leq n$ ($P(1,-1) \equiv \emptyset$); or*
(iv) *$F \sim CP(n) + RP(n)$; or*
(v) *$F \sim RP(n) + CP(r) + CP(n-r-1)$, $0 \leq r \leq n$.*

Moreover, all of these do occur.

(We are indebted to the referee for proving the impossibility of another case for which we had no example, and for the considerable amount of help he has given us toward the format of the paper.)

The proof will proceed via a sequence of lemmas.

LEMMA A. *If for a component F_1, $H^*(F_1)$ is generated by two independent classes of degree one, then either $F \sim RP(1) \times RP(n)$ or $F \sim RP((n-1)\epsilon \bigoplus \sigma)$.*

PROOF. Pick γ and δ using F_1. Then $\gamma^2 = 0$, and so $u = 0$ and γ restricts to v. Pulling (1) back to $F_1 \times BZ/2$ gives $\mathrm{Sq}^1 \delta = \delta t + \gamma\delta$, so $x = 0$ and $F = F_1$.

Let k be maximal with respect to $vy^k \neq 0$. Then v annihilates $H^1(F)$ for $i > k$ and no relation can exist between vy^{i-1} and y^i for $i \leq k$. Since $H^*(F)$ is a Poincaré algebra, [1, 5], $H^i(F) = 0$ for $i > k+1$, $H^{k+1}(F) \cong Z/2$ and y^{k+1} is zero or equal to vy^k. Since $\dim_{Z/2} H^*(F) = 2n+2$, the result follows. $\square$

REMARK. If F_1 and F_2 are two components of F, it can be shown using (1) that γ and δ can be chosen so that their pullbacks to $H^*(F_1 \times BZ/2) \bigoplus H^*(F_2 \times BZ/2)$ must fit one of the following (possibly after interchange of F_1 and F_2):

Case 1. $\gamma^2 = 0$, $\gamma \to (v_1, v_2)$, $\delta \to (v_1 t + z_1, t^2 + z_2)$, $\mathrm{Sq}^1 z_i = v_i z_i$, $v_i^2 = 0$ for $i = 1, 2$, and at least one of v_1, v_2 is nonzero. In this case F_1 and F_2 are the only components of F.

Case 2. $\gamma^2 = \gamma t$, $\gamma \to (0, t)$, $\delta \to (y_1 t + y_1^2, z_2)$.

Case 3. $\gamma^2 = \gamma t$, $\gamma \to (0,0)$, $\delta \to (z_1, t^2 + z_2)$. Since the image of γ in $H^*(F \times BZ/2)$ cannot be zero, this case can only occur when there are three components in F, and the third and first components must then fit Case 2.

LEMMA B. *If $F_1 \sim RP(r)$ or $F_1 \sim CP(r)$, then $r \leq n$.*

PROOF. Suppose $F_1 \sim RP(r)$. Then F has at least two components and if $r > 1$, F_1 and F_2 fit Case 2. There is a relation

$$\delta^{n+1} = \sum_{i=1}^{n+1} (\kappa_i t + \varsigma_i \gamma) t^{2i-1} \delta^{n+1-i} \tag{2}$$

which, in $H^*(F_1 \times BZ/2)$, becomes

$$y_1^{n+1}(y_1 + t)^{n+1} = \sum_{k=0}^{n} \sum_{r=0}^{k} \binom{k}{r} \kappa_{n+1-k} t^{2n+2-2k+r} y_1^{2k-r}.$$

Then $\kappa_{n+1-k} = 0$ for $0 \leq k \leq n$, so $y_1^{n+1} = 0$. $\square$

LEMMA C. *If F has a component F_1 such that $H^*(F_1)$ is generated by classes $a \in H^1$ and $b \in H^2$ with b not a multiple of a^2, then $F \sim P(1, r) + P(1, n - r - 1)$ for $0 \leq r \leq n$.*

PROOF. This is Case 1. Using (2), with Lemma B, and considering the epimorphism $H^*(P(1, r)) \otimes H^*(BZ/2) \to H^*(CP(r)) \otimes H^*(BZ/2)$ (defined algebraically), we see that neither v_1 nor v_2 are zero. $\square$

LEMMA D. *Not all components of F can be $\sim$ to*

(i) *real projective spaces of dimension greater than* 1.

(ii) *complex projective spaces of positive dimension.* $\square$

LEMMA E. *If $F = F_1 + F_2$ and $F_1 \sim RP(1)$, then $F_2 \sim P(1, n - 1)$ or $F_2 \sim CP(1)$ and $n = 1$.* $\square$

LEMMA F. *If $F = F_1 + F_2 + F_3$ and $F_1 \sim RP(r)$ for $r \geq 1$, then $r = n$ and $F_2 + F_3 \sim CP(s) + CP(n - s - 1)$ where $0 \leq s \leq n$.*

PROOF. F_1 with either of the other components must fit Case 2, so $F_2 + F_3 \sim CP(s_2) + CP(s_3)$ and $u_2 = u_3 = 1$. Since u and x are independent we may assume $x_2 = 0$. Then pulling back (2) to $F_2 \times BZ/2$ yields $\kappa_i = \varsigma_i$ for $n + 1 - s_2 \leq i \leq n + 1$. Pulling (2) back to $F_3 \times BZ/2$ then yields $s_3 + 1 \leq n - s_2$, and dimensional requirements show $r = n$ and $s_3 = n - s_2 - 1$. $\square$

LEMMA G. *It is not possible that $F \sim CP(0) + CP(r) + CP(s)$.*

PROOF. Suppose $F \sim CP(0) + CP(r) + CP(s)$. Then $r + s = 2n - 1$ and we may suppose that $r = n$ and $s = n - 1$. Then $y = v = 0$. Pulling (1) back to $F \times BZ/2$ would yield $\chi_2 = 1$, $u_2 = u_3 = 1$ and $\chi_1 = 0$. Since u and x must be independent, we may suppose $x_2 = 0$. Pulling (2) back to $F_2 \times BZ/2$ would yield $\kappa_i = \varsigma_i$ for $1 \leq i \leq n + 1$, so that pulling (2) back to $F_3 \times BZ/2$ would give $(t^2 + z_2)^{n+1} = 0$, which is impossible. $\square$

Finally, we need some examples. Let A denote conjugation in S^1, Q denote conjugation of the (homogeneous) coordinates of $CP(n)$, and B_r denote reflection in the last $n - r$ coordinates of $CP(n)$. These actions induce involutions in $P(1, n)$, with $F_{1 \times Q} = RP(1) \times RP(n)$, $F_{1 \times QB_0} = RP((n-1)\epsilon + \sigma)$, $F_{1 \times B_r} = P(1, r) + P(1, n - r - 1)$, and $F_{A \times B_r} = RP(n) + CP(r) + CP(n - r - 1)$, for $0 \leq r \leq n$.

This completes the proof of Theorem 3. $\square$

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