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CRITERIA FOR REGIONALLY RECURRENT FLOWS

SABER ELAYDI

ABSTRACT. Regionally recurrent and P-regionally recurrent flows are characterized in purely prolongational notions. An example is given to show that the condition $x \in J(x)$ is not, as many authors asserted, equivalent to regional recurrence.

Introduction. Let (X, T) be a flow, where X is a topological space and T is a topological group. Then a point $x \in X$ is said to be T-regionally recurrent or simply regionally recurrent if for each neighbourhood U of x there exists an extensive subset A of T such that $U \cap Ua \neq \emptyset$ for all $a \in A$ [**6**]. In [**3**] the author showed that for a flow (X, T), a point $x \in X$ is regionally recurrent if and only if $x \in J^P(x)$ for all replete semigroups P in T. An example is provided (Example 14) to show that the condition $x \in J(x)$ is not equivalent to regional recurrence as it was stated in [**7**] and [**8**]. However, in the setting of continuous flows, where the phase group is the additive group of real numbers R, a necessary and a sufficient condition for a point $x \in X$ to be regionally recurrent is either one of the following (1) $x \in J^+(x)$ or, (2) $x \in J^-(x)$ or (3) $x \in J(x)$. This is due to the simple structure of replete semigroups in R. Every replete semigroup in R contains either a ray in R^+ or a ray in R^-. Thus if $x \in J^+(x)$, then $x \in J^P(x)$ for all replete semigroups P in R^+. Furthermore, $x \in J^+(x)$ implies that $x \in J^-(x)$. Consequently, $x \in J^Q(x)$ for all replete semigroups Q in R^- and hence x is regionally recurrent. Similar analysis is available for the cases $x \in J^-(x)$ and $x \in J(x)$. The situation is far more complicated in general flows as is shown in Example 14. In this example we have $x \in J(x)$ for each $x \in X$ but $x \notin J^Q(x)$ for some replete semigroups Q in the phase group T.

In [**3**] it was shown that $x \in X$ is P-regionally recurrent iff it is P-nonwandering. The same conclusion holds for T-regionally recurrent and T-nonwandering. Hence, the work here is a natural generalization of the work in [**1**] on nonwandering continuous flows. However, our techniques are different and much simpler.

In addition to the above-mentioned, one purpose of this note is to introduce prolongational techniques which are widely used in Dynamical Systems Theory but surprisingly unknown in the theory of Topological Dynamics. As far as I know it was Hajek [**7**] who suggested the use of prolongational techniques in Topological Dynamics. R. Knight [**8, 9**] and Elaydi [**2–5**], with Kaul followed suit. But the power of these techniques has yet to be shown. Our main references for notations and terminology are [**7**] and [**3**].

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From now on (X,T) denotes a flow, where X is a Hausdorff topological space and T is an abelian topological group. $\mathcal{P}$ and P will denote the set of all replete semigroups in T and a certain replete semigroup in T, respectively.

DEFINITION 1. A subset A of T is P-extensive if $A = S \cap P$ for some extensive set S of T.

LEMMA 2. *The following statements are pairwise equivalent for a subset A of P.*
(1) *A is P-extensive.*
(2) *$A \cap Q \neq \emptyset$ for all $Q \in \mathcal{P}$ and $Q \subset P$.*
(3) *$A \cap pQ \neq \emptyset$ for all $Q \in \mathcal{P}$, $Q \subset P$ and $p \in P$.*

PROOF. (1) implies (2). Since A is P-extensive, $A = S \cap P$ for some extensive set S in T. Hence $S \cap Q \neq \emptyset$ [**6**, 6.15]. This implies that $A \cap Q = S \cap P \cap Q = S \cap Q \neq \emptyset$.

(2) implies (3). Let $p \in P$. Let K be a compact subset of T and let $H = p^{-1}K$. Then there exists $t \in T$ such that $Ht = tp^{-1}K \subset Q$. It follows that $tK \subset pQ$. Hence pQ is a replete semigroup in T which is contained in P. Hence $A \cap pQ \neq \emptyset$.

(3) implies (1). Define $S = A \cap (T - P)$. Then clearly $A = S \cap P$. Furthermore, S is an extensive set in T.

DEFINITION 3. A point $x \in X$ is said to be P-regionally recurrent if for each neighbourhood U of x there exists a P-extensive set A such that $U \cap Ua \neq \emptyset$ for all $a \in A$.

A point $x \in X$ is T-regionally recurrent or simply regionally recurrent if it is P-regionally recurrent for all $P \in \mathcal{P}$. A transformation group (X,T) is said to have one of the above properties if each point in X has the property.

THEOREM 4. *For any flow (X,T) the following statements are pairwise equivalent.*
(1) *The flow is P-regionally recurrent.*
(2) *$x \in J^Q(x)$ for all $Q \in \mathcal{P}$, $Q \subset P$ and $x \in X$.*
(3) *$J^Q(x) = D^Q(x)$ for all $Q \in \mathcal{P}$, $Q \subset P$ and $x \in X$.*
(4) *$D^Q(x) = D^{Q^{-1}}(x)$ for all $Q \in \mathcal{P}$, $Q \subset P$ and $x \in X$.*

PROOF. (1) implies (2). Let $Q \in \mathcal{P}$ such that $Q \subset P$ and let U be a neighbourhood of $x \in X$. Then $U \cap Ua \neq \emptyset$ for all $a \in A$, for some p-extensive set A. Then by Lemma 2 for each $q \in Q$ there exists $a \in A \cap qQ$. Hence $x \in \overline{UqQ}$ for each $q \in Q$. This implies that $x \in J^Q(x)$.

(2) implies (3). Since $J^Q(x)$ is invariant, $xQ \subset J^Q(x)$. Hence $D^Q(x) = xQ \cup J^Q(x) = J^Q(x)$.

(3) implies (4). Let $y \in D^{Q^{-1}}(x)$. Then $x \in D^Q(y) = J^Q(y)$. This implies that $y \in J^{Q^{-1}}(x)$. Thus $J^{Q^{-1}}(x) = D^{Q^{-1}}(x)$. Let $z \in D^Q(x) = J^Q(x)$. Then $zQ^{-1} \subset J^Q(x)$. If $z \notin J^{Q^{-1}}(x)$, then $zQ^{-1} \cap J^{Q^{-1}}(x) = zQ^{-1} \cap D^{Q^{-1}}(x) = \emptyset$ which is clearly absurd. Hence $D^Q(x) = D^{Q^{-1}}(x)$.

(4) implies (1). Let $x \in X$ and $Q \in \mathcal{P}$ with $Q \subset P$. Since $xQ \subset D^Q(x) = D^{Q^{-1}}(x)$, for each $q \in Q$ we have $xq^{-1} \in D^Q(x) = D^{Q^{-1}}(x)$. This implies that $x \in D^{Q^{-1}}(x)q = D^{Q^{-1}}(xq) = D^Q(xq)$ for all $q \in Q$. Thus for each neighbourhood U of x and $q \in Q$ we have $x \in \overline{UqQ}$. For each $q \in Q$ choose $b \in qQ$ such that $U \cap Ub \neq \emptyset$. Let A be the set of all such elements b with the above-mentioned properties ranging over all replete semigroups Q contained in P. Then according to

Lemma 2 the set A is P-extensive. Consequently x is P-regionally recurrent. This completes the proof of the theorem.

COROLLARY 5. *For any flow* (X, T) *the following statements are pairwise equivalent.*

(1) *The flow* (X, T) *is regionally recurrent.*
(2) $x \in J^P(x)$ *for each* $x \in X$ *and* $P \in \mathcal{P}$.
(3) $J^P(x) = D^P(x)$ *for each* $x \in X$ *and* $P \in \mathcal{P}$.
(4) $D^P(x) = D^{P^{-1}}(x)$ *for each* $x \in X$ *and* $P \in \mathcal{P}$.

LEMMA 6. *Let* $\{x_i\}$ *and* $\{y_i\}$ *be nets in* X *such that* $x_i \to x$ *and* $y_i \to y$. *If for each* i, $y_i \in J^P(x_i)$ *then* $y \in J^P(x)$.

PROOF. The proof follows immediately from [**2**, 2.7].

THEOREM 7. *A flow* (X, T) *is p-regionally recurrent iff the set of p-regionally recurrent points in* X *is dense in* X.

PROOF. This follows from Lemma 6 and Theorem 4.

COROLLARY 8. *A flow* (X, T) *is regionally recurrent iff the set of regionally recurrent points in* X *is dense in* X.

LEMMA 1.9. *If* $y \in L^P(x)$ *for* $P \in \mathcal{P}$, *then* $J^P(x) \subset J^P(y) = D^P(y)$.

PROOF. Let $z \in J^P(x)$. Since $J^P(xp) = J^P(x)$ for all $p \in \mathcal{P}$, $z \in J^P(xp)$ for each $p \in P$. There exists a net $xp_i \to y$, $p_i \in P$. It follows from Lemma 6 that $z \in J^P(y)$. Hence $J^P(x) \subset J^P(y)$. Furthermore, $y \in L^P(x) \subset J^P(x) \subset J^P(y)$ implies that $J^P(y) = D^P(y)$, since $D^P(y) = yP \cup J^P(y) = J^P(y)$, as $J^P(y)$ is invariant.

THEOREM 10. *If* $y \in L^Q(x)$ *for all* $Q \in \mathcal{P}$ *contained in* P, *then* y *is p-regionally recurrent.*

PROOF. Use Lemma 9 and Theorem 4.

COROLLARY 11. *If* $y \in L^P(x)$ *for all* $P \in \mathcal{P}$, *then* y *is regionally recurrent.*

THEOREM 12. *Let* (X, T) *be a* P*-regionally recurrent flow. Then the set of* P*-recurrent points in* X *is dense in* X, *provided that* T *is generative* [**6**].

PROOF. We need only to replace T by Q, where Q is any replete semigroup contained in P, in the proof of [**8**, 1].

REMARK 1.13. Although the proof of [**8**, 1] uses wrong equivalences to regionally recurrence and recurrence it can be corrected by a slight modification. We will give an example of a flow in which $x \in L(x)$ for each $x \in X$ and yet it is neither regionally recurrent nor recurrent.

EXAMPLE 14. Let (X, T) be a flow, where $X = R$; the set of real numbers with the usual topology and $T = R \times R$, with the operation $(a, b) + (c, d) = (a + c, b + d)$, where R is the additive group of real numbers. Define the action of T on X by $x(a, b) = x + a + b$ for all $x \in X$ and $(a, b) \in R \times R$. Let $P = \{(x, y) \in R \times R | x \geq 0, y \leq 0\}$. Then for each $x \in X$, $x \in L^P(x) \cap L^{P^{-1}}(x)$. Let $Q = \{(x, y) \in R \times R | x \geq 0, y \geq 0\}$. Then for each $x \in X$ we have $x \notin L^Q(x) =$

$J^Q(x)$. Hence the flow is not regionally recurrent. Furthermore, $x \notin L^S(x)$ for each $x \in X$, where $S = \{(x,y) \in R \times R | x \geq 0, -x/3 \leq y \leq 0\}$ is a replete semigroup contained in P. Thus the flow is not even P-regionally recurrent.

REMARK 15. In the above example the flow is of P-characteristic 0 [**2**]. Also the example [**5**, 5.1] is of P-characteristic 0, where $P = R^+$, but not P-regionally recurrent. In the sequel we will discuss a few simple relations between these two notions.

THEOREM 16. *Let (X,T) be a flow of P-characteristic* 0. *If $L^{Q^{-1}}(x) \neq \emptyset$ for all $x \in X$ and $Q \in \mathcal{P}$ contained in P, then (X,T) is P-regionally recurrent.*

PROOF. Use [**2**, 3.2] and Theorem 4.

COROLLARY 17. *Let (X,T) be a flow of P-characteristic* 0 *where $L^{P^{-1}}(x) \neq \emptyset$ for all $x \in X$ and $P \in \mathcal{P}$. Then (X,T) is regionally recurrent.*

THEOREM 18. *Let (X,T) be a flow, where X is assumed to be locally compact and T is generative. Then (X,T) is regionally recurrent and of P-characteristic* 0 *for all $P \in \mathcal{P}$ iff it is pointwise almost periodic and of Q-characteristic* 0 *for some $Q \in \mathcal{P}$.*

PROOF. Assume that (X,T) is regionally recurrent and of P-characteristic 0 for all $P \in \mathcal{P}$. Then it follows from [**2**, 3.4] and Corollary 1.5 that $x \in L^P(x)$ for each $x \in X$ and $P \in \mathcal{P}$. Hence the flow (X,T) is recurrent [**2**, 3.7]. Now use [**6**, 7.05] and [**2**, 3.5] to conclude the proof of the first part. To prove the converse assume that the flow is of Q-characteristic 0 for some $Q \in \mathcal{P}$ and pointwise almost periodic. Then $\overline{xT}$ is compact and minimal for all $x \in X$ [**6**, 4.10]. We need to show that the flow is of P-characteristic 0 for each $P \in \mathcal{P}$. Let $y \in D^P(x)$, x, $y \in X$ and $P \in \mathcal{P}$. Then there are nets $\{x_i\}$ in X and $\{p_i\}$ in P such that $x_i \to x$ and $x_i p_i \to y$. For each i, $x_i p_i \in \overline{x_i P} \subset L^{Q^{-1}}(x) \subset D^{Q^{-1}}(x)$. It follows from [**2**, 2.7] that $y \in D^{Q^{-1}}(x) = D^Q(x) = \overline{xQ} = \overline{xT} = \overline{xP}$. This completes the proof of the theorem.

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DEPARTMENT OF MATHEMATICS, KUWAIT UNIVERSITY, P.O. BOX 5969, KUWAIT