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A GENERALIZATION OF THE COHOMOLOGY OF GROUPS

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ABSTRACT. Generalizations of the cohomology of finite groups, in which one considers varying families of subgroups, are presented. These groups are shown to relate to Bredon equivariant of homology of universal G-spaces, and to lead to necessary algebraic conditions for G-actions on contractible spaces.

1. Introduction. In this note, we present new algebraic invariants of a finite group G generalizing the notion of cohomology of G. Specifically, given a collection $\mathcal{F}$ of subgroups of G closed under conjugation, as well as cohomological ("Hecke") functors T and T' in the sense of [**G1**], we construct groups $H^*((T,\mathcal{F});T')$ with the property that $H^*((\hat{\mathbf{Z}},\{1\});\hat{A}) \cong H^*(G;A)$ for any $\mathbf{Z}G$-module A, where, for any $\mathbf{Z}G$-module B, $\hat{B}$ denotes the associated Hecke functor $\hat{B}(G/H) = B^H$. (The notion of a Hecke functor will be reviewed in §2.) The potential for such generalizations is alluded to in [**RS**].

As a topological application, we show in §3 that $H^*((T,\mathcal{F});T')$ represents the equivariant Bredon cohomology of an associated universal G-space when $T = \mathbf{Z}$. (The constructions in §2 therefore give one an explicit computational machine for these cohomology groups.) In addition, we show that if G acts on a suitable finite contractible complex with specified orbit types, then $H^*((T,\mathcal{F});T')$ must vanish above the dimension of the complex.

The author is grateful to Professor Leonard Scott for his suggestions, particularly for his novel definition of a Hecke functor.

2. Mackey functors and the Bar resolution. First, we set up a few categories. Denote by $\mathcal{G}$ the category whose objects are the G-sets G/H with $H \subset G$ and whose morphisms are the equivariant maps. Thus a morphism $f\colon G/K \to G/H$ must have the form $f(gK) = gg'H$, where $K \subset g'Hg'^{-1}$. $\mathbf{Z}\mathcal{G}$ will denote the category whose objects are those of $\mathcal{G}$ but whose morphisms $G/K \to G/H$ are the $\mathbf{Z}G$-module homomorphisms $\mathbf{Z}G/K \to \mathbf{Z}G/H$, where $\mathbf{Z}G/J$ denotes the free $\mathbf{Z}$-module on the G-set G/J. If $\mathcal{F}$ is a family of subgroups of G closed under conjugation, then we may define associated categories $\mathcal{G}(\mathcal{F})$ and $\mathbf{Z}\mathcal{G}(\mathcal{F})$ by considering, respectively, the full subcategories of $\mathcal{G}$ and $\mathbf{Z}\mathcal{G}$ whose objects are those G/H with $H \in \mathcal{F}$.

Recall that a coefficient system in the sense of Bredon [**B1**] is a contravariant functor $T\colon \mathcal{G} \to \mathcal{A}b$, where $\mathcal{A}b$ is the category of abelian groups. A Hecke functor is usually thought of as a bifunctor $(T^*, T_*)\colon \mathcal{G} \to \mathcal{A}b$ where T^* is contravariant and T_* is covariant, obeying certain axioms (see [**G1**]). Following is what I believe to be the most succinct definition of a Hecke functor, suggested to me by Leonard Scott.

DEFINITION 2.1. A Hecke functor is an additive contravariant functor

$$T\colon \mathbf{Z}\mathcal{G} \to \mathcal{A}b.$$

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More generally, an $\mathcal{F}$-Hecke functor is an additive contravariant functor

$$T\colon \mathbf{Z}\mathcal{G}(\mathcal{F}) \to \mathcal{A}b.$$

The collection of $\mathcal{F}$-Hecke functors forms a category whose morphisms are the natural transformations.

REMARKS 2.2. The natural inclusion $\mathcal{G} \to \mathbf{Z}\mathcal{G}$ displays every Hecke functor as, in particular, a coefficient system. Further, the inclusion $\mathbf{Z}N(H) \to \mathbf{Z}\mathcal{G}(G/H, G/H)$ displays each $T(G/H)$ as a $\mathbf{Z}N(H)$-module, whose $N(H)$ denotes the normalizer of $H \subset G$. It is not hard to show that our definition of a Hecke functor is equivalent to Green's. It is, however, stronger than the notion of a Mackey functor as in, for example, [**D1**].

EXAMPLES OF HECKE FUNCTORS 2.3. If R is a $\mathbf{Z}G$-module, one obtains an associated $\mathcal{F}$-Hecke functor, $\hat{R}$, as follows. If $H \in \mathcal{F}$, take $\hat{R}(G/H) = R^H$, and if $f\colon \mathbf{Z}G/K \to \mathbf{Z}G/H$ is specified by $f(eK) = \sum n_i g_i H$, one may take $\hat{R}(f)\colon \hat{R}(G/H) \to \hat{R}(G/K)$ to be given by $\hat{R}(f)(a) = \sum n_i g_i a$. In particular, if R is a trivial $\mathbf{Z}G$-module, one obtains the constant $\mathcal{F}$-Hecke functor $\hat{R}(G/H) = R$ for each $H \in \mathcal{F}$.

CONSTRUCTION 2.4. Fix an $\mathcal{F}$-Hecke functor T, and denote $\mathbf{Z}\mathcal{G}(\mathcal{F})$ by $\mathcal{C}$ for brevity. Let $W_*(T, \mathcal{F})$ be the d.g. $\mathbf{Z}G$-module given by

$$W_n(T,\mathcal{F}) = \bigoplus \mathbf{Z}G/H_0 \otimes [\mathcal{C}(G/H_0, G/H_1) \otimes \cdots \otimes \mathcal{C}(G/H_{n-1}, G/H_n)] \otimes T(G/H_n),$$

where the sum is taken over all distinct sequences $(H_0, \dots, H_n)$ of subgroups in $\mathcal{F}$. The action of $\mathbf{Z}G$ is the natural left action, and we define $\mathbf{Z}G$-homomorphisms $d_n\colon W_n(T,\mathcal{F}) \to W_{n-1}(T,\mathcal{F})$ by $d_n(x) = \sum_{i=0}^n (-1)^i F_i(x)$, where F_i is given on generators by

$$F_i(h[f_1, \dots, f_n]t) = \begin{cases} f_1(y)[f_2, \dots, f_n]t & \text{if } i = 0; \\ y[f_1, \dots, f_{n-1}]f_n^*(t) & \text{if } i = n; \\ y[f_1, \dots, f_i f_{i+1}, \dots, f_n]t & \text{otherwise.} \end{cases}$$

That d_* is indeed a differential of this graded $\mathbf{Z}G$-module is easy to check.

Construction 2.4 may be thought of as a two-sided algebraic Bar construction $B_*(\mathcal{O}, \mathbf{Z}\mathcal{G}(\mathcal{F}), T)$ analogous to Elmendorf's geometric construction in [**E1**]. Here, $\mathcal{O}$ denotes the object space of $\mathbf{Z}\mathcal{G}(\mathcal{F})$, and the general construction is fully explained in [**E1**].

We consider now the basic properties of $W_*(T, \mathcal{F})$.

PROPOSITION 2.5. *Let $H \in \mathcal{F}$. Then $(W_*(T,\mathcal{F})^H, d_*^H)$ is a resolution of $T(G/H)$ by $\mathbf{Z}N(H)$-modules.*

PROOF. We have a natural isomorphism

$$\sigma\colon W_*(T,\mathcal{F})^H \cong \mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, W_*(T,\mathcal{F}))$$

of $\mathbf{Z}N(H)$-modules, specified by sending x to the unique $\mathbf{Z}G$-map $f\colon \mathbf{Z}G/H \to W_*(T,\mathcal{F})$ specified by $f(eH) = x$.

Further,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, \mathbf{Z}G/K \otimes M) &\cong \mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, \mathbf{Z}G/K) \otimes M \\ &= \mathbf{Z}\mathcal{G}(\mathcal{F})(G/H, G/K) \otimes M \end{aligned}$$

for any trivial $\mathbf{Z}G$-module M. This gives

$$W_n(T,\mathcal{F})^H \cong \bigoplus \mathcal{C}(G/H, G/H_0) \otimes [\mathcal{C}(G/H_0, G/H_1) \otimes \cdots \otimes \mathcal{C}(G/H_{n-1}, G/H_n)] \otimes T(G/H_n),$$

with the notation of 2.4. Under this identification, define ϵ: $W_0(T,\mathcal{F})^H \to T(G/H)$ by taking a summand $f[\,]t$ to $f^*(t)$. A contracting chain homotopy is then given by

$$s_n(f[f_1,\ldots,f_n]t) = (1_H[f, f_1, \ldots, f_n]t)$$

where 1_H is the identity morphism $G/H \to G/H$. It is now formal that $ds - sd = 1$. □

REMARKS 2.6. (i) Note that $W_*(T,\mathcal{F})^H$ is not acyclic as a $\mathbf{Z}N(H)$-module; the contracting homotopy does not respect the $N(H)$-action.

(ii) If $\mathcal{F} = \{1\}$ and $T = \mathbf{Z}$, then $W_*(T,\mathcal{F})$ is a free $\mathbf{Z}G$-resolution of the trivial G-module $\mathbf{Z}$, and is, in fact, isomorphic with the usual Bar resolution.

(iii) $\mathbf{Z}G/H$ is not, in general, a projective $\mathbf{Z}G$-module. It is therefore inappropriate to view $W_*(T,\mathcal{F})$ as a resolution in the category of $\mathbf{Z}G$-modules.

Regard $W_*(T,\mathcal{F})$ as an $\mathcal{F}$-Hecke functor as in 2.3; we set $W_*(T,\mathcal{F})(G/H) = W_*(T,\mathcal{F})^H \cong \mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, W_*(T,\mathcal{F}))$. (Note that this isomorphism demonstrates the action of $\mathbf{Z}\mathcal{G}(\mathcal{F})$ as the obvious left one.)

PROPOSITION 2.7. *Let $H \in \mathcal{F}$. Then $\mathbf{Z}G/H$ is a projective $\mathcal{F}$-Hecke functor.*

PROOF. Denote the category of $\mathcal{F}$-Hecke functors by $\mathcal{M}(\mathcal{F})$. We now claim that $\mathcal{M}(\mathcal{F})(\mathbf{Z}G/H, T) \cong T(G/H)$, naturally in T, for every $T \in \mathcal{M}(\mathcal{F})$. Indeed, define ϕ: $\mathcal{M}(\mathcal{F})(\mathbf{Z}G/H, T) \to T(G/H)$ by taking f to $f(G/H)(eH)$ and ψ: $T(G/H) \to \mathcal{M}(\mathcal{F})(\mathbf{Z}G/H, T)$ by taking $\psi(t)(G/K)$ to be the composite

$$(\mathbf{Z}G/H)^K \overset{\sigma}{\cong} \mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/K, \mathbf{Z}G/H) \overset{\epsilon}{\to} T(G/K),$$

where $\epsilon(r) = T(r)(t)$. Thus $\psi(t)(G/K)(x) = T(\sigma(x))(t)$. That $\psi(t)$ is indeed a morphism in $\mathcal{M}(\mathcal{F})$ follows by the naturality of the definition of ψ, and that $\phi\psi = 1$ is clear. That $\psi\phi$ is the identity is a consequence of the following. Given any morphism f: $\mathbf{Z}G/H \to T$ in $\mathcal{M}(\mathcal{F})$ and $x \in (\mathbf{Z}G/H)^K$ with $K \in \mathcal{F}$, the diagram

$$\begin{array}{ccc} (\mathbf{Z}G/H)^H & \overset{f(G/H)}{\longrightarrow} & T(G/H) \\ {\scriptstyle \sigma(x)^*}\downarrow & & \downarrow {\scriptstyle T(\sigma(x))} \\ (\mathbf{Z}G/H)^K & \overset{f(G/K)}{\longrightarrow} & T(G/K) \end{array}$$

must commute, where $\sigma(x)^*$ is induced by $\sigma(x)$: $\mathbf{Z}G/K \to \mathbf{Z}G/H$; $eK \mapsto x$. Chasing eH around the diagram gives

$$f(G/K)(\sigma(x)^* eH) = f(G/K)(x) = T(\sigma(x))(f(G/H)(eH)),$$

showing that $f(G/K)(x)$ is completely specified by $f(G/H)(eH)$ via application of $T(\sigma(x))$. This shows that $\psi\phi = 1$ as claimed.

For projectivity of $\mathbf{Z}G/H$ as an object in $\mathcal{M}(\mathcal{F})$, we now observe that one may complete any diagram of the form

$$\begin{array}{ccc} & {\scriptstyle g}\nearrow & T' \\ & & \downarrow {\scriptstyle p} \\ \mathbf{Z}G/H & \underset{f}{\longrightarrow} & T \end{array}$$

in $\mathcal{M}(\mathcal{F})$ by setting $g = \psi(t)$ where $t \in T'(G/H)$ is any element in $p^{-1}(f(G/H)(eH))$. □

REMARKS 2.8. If $H \notin \mathcal{F}$, then the assertion is false; let $\mathcal{F} = \{1\}$ and $H = G$. Then $\mathbf{Z}G/G = \mathbf{Z}$ is not a projective $\mathbf{Z}G$-module. Hence it is not projective in $\mathcal{M}(\mathcal{F})$, as one notes that if $\mathcal{F} = \{1\}$, $\mathcal{M}(\mathcal{F})$ is equivalent to the category of $\mathbf{Z}G$-modules. $\mathbf{Z}$ is, however, a projective $\mathcal{M}(\{1, g\})$-module by the proposition.

If each $T(G/H)$ is a projective $\mathbf{Z}$-module, then it follows that $W_*(T, \mathcal{F})$ is a projective resolution of T in the category $\mathcal{M}(\mathcal{F})$. (One should not expect the Bar resolution to yield projective modules in general—even in the case G trivial.) As far as the applications are concerned, we may take each $T(G/H)$ to be projective. In general, there are "enough" projectives; if $T \in \mathcal{M}(\mathcal{F})$, and if $F(G/H)$ is the free abelian group on the elements of $T(G/H)$, then the map

$$\epsilon\colon \bigoplus_{H \in \mathcal{F}} \mathbf{Z}G/H \otimes F(G/H) \to T$$

of $\mathcal{F}$-Mackey functors, defined just as in the proof of Proposition 2.5, is an epimorphism in $\mathcal{M}(\mathcal{F})$.

Now let M_* be any d.g. $\mathcal{F}$-Hecke functor, and let $T \in \mathcal{M}(\mathcal{F})$.

DEFINITION 2.9. The cohomology of M_* with T-coefficients, $H^*(M_*; T)$, is defined to be $H^*(\mathcal{M}(\mathcal{F})(M_*, T))$, (where, as usual, $\mathcal{M}(\mathcal{F})$(-,-) denotes Hom in the category $\mathcal{M}(\mathcal{F})$).

EXAMPLES 2.10. (i) Let X be any G-CW complex with cells of type $G/H \times D^n$ with $H \in \mathcal{F}$. Then the cellular chain complex, $C_*(X)$ is a d.g. $\mathcal{F}$-Hecke functor if we define $C_*(X)(G/H) = C_*(X)^H$ ($\neq C_*(X^H)$). Since $C_*(X)$ is a sum of G-modules $\mathbf{Z}G/H$ with $H \in \mathcal{F}$, it is a projective $\mathcal{F}$-Hecke functor by Proposition 2.7. We shall see in §3 that $H^*(C_*(X); T)$ is just Bredon cohomology, $H^*_G(X; T)$, as in [**B1**], provided $\mathcal{F}$ has the property that $H \in \mathcal{F}$, $K \subset H$ implies $K \in \mathcal{F}$. (This condition also allows one to think of T as a coefficient system defined on all orbit-types, so that $H^*_G(\text{-}; T)$ is meaningful.)

(ii) If $M_* = W_*(T, \mathcal{F})$ with $\mathcal{F} = \{1\}$ and $T = \hat{\mathbf{Z}}$, it follows that $H^*(M_*; \hat{R}) \cong H^*(G; R)$ for all $\mathbf{Z}G$-modules R.

(iii) In general, define $H^*((T, \mathcal{F}); T')$ to be $H^*(W_*(T, \mathcal{F}); T')$. This generalizes the notion of $H^*(G; R)$ to larger families of subgroups, and is independent of the choice of resolution $W_*(T, \mathcal{F})$ if each $T(G/H)$ is projective.

3. Applications. Let $\mathcal{F}$ be a family of subgroups with the property that, if $H \in \mathcal{F}$ and $gKg^{-1} \subset H$ for some $g \in G$, then $K \in \mathcal{F}$.

By a universal G-space $E\mathcal{F}$, we shall mean a G-space with $E\mathcal{F}^H$ contractible for each $H \in \mathcal{F}$, and $E\mathcal{F}^H = \emptyset$ otherwise. (For example, $E\{1\} = EG$, the standard universal G-space.) These spaces were first constructed by Palais, while Elmendorf has an elegant construction in terms of the geometric two-sided Bar construction in [**E1**]. As a consequence of these constructions, we may assume that $E\mathcal{F}$ is a G-CW complex with cells of type $G/H \times D^n$ and $H \in \mathcal{F}$.

If T is in $\mathcal{M}(\mathcal{F})$, we consider the problem of computing $H^*_G(E\mathcal{F}; T)$, the Bredon cohomology of $E\mathcal{F}$. (As a special case, $H^*_G(EG; \hat{R}) \cong H^*(G; R)$.)

As promised in §2, we have the following result.

LEMMA 3.1. *Let X be any G-CW complex with cells of type $G/H \times D^n$ with $H \in \mathcal{F}$. Then $H^*(C_*(X); T) \cong H^*_G(X; T)$, Bredon cohomology of X with coefficients in the coefficient system T, where we take $T(G/K) = 0$ if $K \notin \mathcal{F}$.*

PROOF. It suffices to construct a natural transformation

$$\psi\colon H^*(C_*(X);T) \to H^*_G(X;T)$$

which restricts to an isomorphism where $X = G/H$ and $H \in \mathcal{F}$, and to verify that $H^*(C_*(X);T)$ is indeed a generalized equivariant cohomology theory in the sense of Bredon.

The latter claim is clear as a consequence of the projectivity of $C_*(X)$. For the former, recall from [**B1**] that $H^*_G(X;T)$ is the cohomology of $\mathrm{Hom}(\underline{C}_*(X),T)$ in the category of coefficient systems, where $\underline{C}_*(X)(G/H) = C_*(X^H)$ for all $H \subset G$. Since $X^H = \emptyset$ for $H \notin \mathcal{F}$, it follows that $\underline{C}_*(X)(G/H) = 0$ for such H. Now define $\psi\colon \mathcal{M}(\mathcal{F})(C_*(X),T) \to \mathrm{Hom}(\underline{C}_*(X),T)$ by

$$f(G/H) \mapsto \begin{cases} f(G/H)|C_*(X^H) & \text{if } H \in \mathcal{F}; \\ 0 & \text{if } H \notin \mathcal{F}. \end{cases}$$

Then ψ induces the desired transformation after passage to cohomology. Further, if $H \in \mathcal{F}$, then $\mathcal{M}(\mathcal{F})(C_*(G/H),T) \cong \mathcal{M}(\mathcal{F})(\mathbf{Z}G/H,T) \cong T(G/H)$, and under this isomorphism, ψ becomes the identity. □

PROPOSITION 3.2. *There exists an isomorphism, natural in T,*

$$\phi\colon H^*_G(E\mathcal{F};T) \cong H^*((\hat{\mathbf{Z}},\mathcal{F});T).$$

PROOF. By the lemmas,

$$H^*_G(E\mathcal{F};T) \cong H^*(C_*(E\mathcal{F});T), \text{ and } C_*(E\mathcal{F})^H \cong C_*(E\mathcal{F}/H)$$

for all $H \in \mathcal{F}$. Further, since $E\mathcal{F}^K$ is contractible for each $K \subset H$, it follows that $E\mathcal{F}$ is H-equivariantly contractible, whence $E\mathcal{F}/H$ is contractible. Thus $C_*(E\mathcal{F})$ is a projective resolution of $\hat{\mathbf{Z}}$ in $\mathcal{M}(\mathcal{F})$, and the result now follows by the lemma. □

REMARKS 3.3. Proposition 3.2 permits explicit computation of $H^*_G(E\mathcal{F};T)$ since one can write down the appropriate Bar resolution $W_*(T,\mathcal{F})$. It also follows that $H^*_G(E\mathcal{F};T)$ is an algebraic invariant of G, $\mathcal{F}$ and T. Of course, when T is replaced by $\hat{R}$ and $\mathcal{F}$ by $\{1\}$, we obtain the expected isomorphism $H^*_G(EG;\hat{R}) \cong H^*(G;R)$.

For the next application, we permit $\mathcal{F}$ to denote any family of subgroups closed under conjugation.

PROPOSITION 3.4. *Let X be a G-*CW *complex of dimension n such that X/H is $\mathbf{Z}$-acyclic for each $H \in \mathcal{F}$, and such that X has all orbits of the form G/H with $H \in \mathcal{F}$. Then $H^p((\hat{\mathbf{Z}},\mathcal{F});T) = 0$ for all $p > n$.*

PROOF. $C_*(X)$ is, under the hypothesis, a projective resolution of $\hat{\mathbf{Z}}$ in $\mathcal{M}(\mathcal{F})$, where $H^p(C_*(X);T) \cong H^p((\hat{\mathbf{Z}},\mathcal{F});T)$ for all p. Since the former group vanishes for $p > n$, the result follows. □

COROLLARY 3.5. *Let G act on a space X such that*
(i) *X inherits the structure of a contractible G-*CW *complex*;
(ii) $\dim X = n$;
(iii) *X has orbit-types G/H with $H \in \mathcal{F}$.*
Then $H^p((\hat{\mathbf{Z}},\mathcal{F});T) = 0$ for all $p > n$.

PROOF. By [**B2**, 5.4], $C_*(X/H)$ is $\mathbf{Z}$-acyclic for each $H \subset G$. The result now follows by 3.4. □

COROLLARY 3.6. *Let $G \in \mathcal{F}$. Then*

$$H^p((\hat{\mathbf{Z}}, \mathcal{F}); T) \cong \begin{cases} 0 & \text{if } p > 0, \\ T(G/G) & \text{if } p = 0 \end{cases}$$

for all T.

PROOF. We apply Corollary 3.5 with $X =$ point, obtaining $H^p((\hat{\mathbf{Z}}, \mathcal{F}); T) = 0$ for $p > 0$. The case $p = 0$ is then obtained by writing out $C*$(point) and observing that $\mathcal{M}(\mathcal{F})(\hat{\mathbf{Z}}, T) \cong T(G/G)$, as in the proof of 2.7. □

Corollary 3.5 gives the lowest possible dimension of a finite contractible CW-complex X with a suitable G-action. This problem has been examined by Oliver and Assadi (see [**A1**]), and they obtain necessary and sufficient conditions on the subgroups of G for such actions to exist.

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