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A GENERALIZATION OF THE COHOMOLOGY OF GROUPS

STEFAN WANER

ABSTRACT. Generalizations of the cohomology of finite groups, in which one considers varying families of subgroups, are presented. These groups are shown to relate to Bredon equivariant of homology of universal G-spaces, and to lead to necessary algebraic conditions for G-actions on contractible spaces.

1. Introduction. In this note, we present new algebraic invariants of a finite group G generalizing the notion of cohomology of G. Specifically, given a collection $\mathcal{F}$ of subgroups of G closed under conjugation, as well as cohomological ("Hecke") functors T and T' in the sense of [**G1**], we construct groups $H^*((T,\mathcal{F});T')$ with the property that $H^*((\hat{\mathbf{Z}},\{1\});\hat{A}) \cong H^*(G;A)$ for any $\mathbf{Z}G$-module A, where, for any $\mathbf{Z}G$-module B, $\hat{B}$ denotes the associated Hecke functor $\hat{B}(G/H) = B^H$. (The notion of a Hecke functor will be reviewed in §2.) The potential for such generalizations is alluded to in [**RS**].

As a topological application, we show in §3 that $H^*((T,\mathcal{F});T')$ represents the equivariant Bredon cohomology of an associated universal G-space when $T = \mathbf{Z}$. (The constructions in §2 therefore give one an explicit computational machine for these cohomology groups.) In addition, we show that if G acts on a suitable finite contractible complex with specified orbit types, then $H^*((T,\mathcal{F});T')$ must vanish above the dimension of the complex.

The author is grateful to Professor Leonard Scott for his suggestions, particularly for his novel definition of a Hecke functor.

2. Mackey functors and the Bar resolution. First, we set up a few categories. Denote by $\mathcal{G}$ the category whose objects are the G-sets G/H with $H \subset G$ and whose morphisms are the equivariant maps. Thus a morphism $f\colon G/K \to G/H$ must have the form $f(gK) = gg'H$, where $K \subset g'Hg'^{-1}$. $\mathbf{Z}\mathcal{G}$ will denote the category whose objects are those of $\mathcal{G}$ but whose morphisms $G/K \to G/H$ are the $\mathbf{Z}G$-module homomorphisms $\mathbf{Z}G/K \to \mathbf{Z}G/H$, where $\mathbf{Z}G/J$ denotes the free $\mathbf{Z}$-module on the G-set G/J. If $\mathcal{F}$ is a family of subgroups of G closed under conjugation, then we may define associated categories $\mathcal{G}(\mathcal{F})$ and $\mathbf{Z}\mathcal{G}(\mathcal{F})$ by considering, respectively, the full subcategories of $\mathcal{G}$ and $\mathbf{Z}\mathcal{G}$ whose objects are those G/H with $H \in \mathcal{F}$.

Recall that a coefficient system in the sense of Bredon [**B1**] is a contravariant functor $T\colon \mathcal{G} \to \mathcal{A}b$, where $\mathcal{A}b$ is the category of abelian groups. A Hecke functor is usually thought of as a bifunctor $(T^*, T_*)\colon \mathcal{G} \to \mathcal{A}b$ where T^* is contravariant and T_* is covariant, obeying certain axioms (see [**G1**]). Following is what I believe to be the most succinct definition of a Hecke functor, suggested to me by Leonard Scott.

DEFINITION 2.1. A Hecke functor is an additive contravariant functor

$$T\colon \mathbf{Z}\mathcal{G} \to \mathcal{A}b.$$

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More generally, an $\mathcal{F}$-Hecke functor is an additive contravariant functor

$$T\colon \mathbf{Z}\mathcal{G}(\mathcal{F}) \to \mathcal{A}b.$$

The collection of $\mathcal{F}$-Hecke functors forms a category whose morphisms are the natural transformations.

REMARKS 2.2. The natural inclusion $\mathcal{G} \to \mathbf{Z}\mathcal{G}$ displays every Hecke functor as, in particular, a coefficient system. Further, the inclusion $\mathbf{Z}N(H) \to \mathbf{Z}\mathcal{G}(G/H, G/H)$ displays each $T(G/H)$ as a $\mathbf{Z}N(H)$-module, whose $N(H)$ denotes the normalizer of $H \subset G$. It is not hard to show that our definition of a Hecke functor is equivalent to Green's. It is, however, stronger than the notion of a Mackey functor as in, for example, [**D1**].

EXAMPLES OF HECKE FUNCTORS 2.3. If R is a $\mathbf{Z}G$-module, one obtains an associated $\mathcal{F}$-Hecke functor, $\hat{R}$, as follows. If $H \in \mathcal{F}$, take $\hat{R}(G/H) = R^H$, and if $f\colon \mathbf{Z}G/K \to \mathbf{Z}G/H$ is specified by $f(eK) = \sum n_i g_i H$, one may take $\hat{R}(f)\colon \hat{R}(G/H) \to \hat{R}(G/K)$ to be given by $\hat{R}(f)(a) = \sum n_i g_i a$. In particular, if R is a trivial $\mathbf{Z}G$-module, one obtains the constant $\mathcal{F}$-Hecke functor $\hat{R}(G/H) = R$ for each $H \in \mathcal{F}$.

CONSTRUCTION 2.4. Fix an $\mathcal{F}$-Hecke functor T, and denote $\mathbf{Z}\mathcal{G}(\mathcal{F})$ by $\mathcal{C}$ for brevity. Let $W_*(T, \mathcal{F})$ be the d.g. $\mathbf{Z}G$-module given by

$$W_n(T,\mathcal{F}) = \bigoplus \mathbf{Z}G/H_0 \otimes [\mathcal{C}(G/H_0, G/H_1) \otimes \cdots \otimes \mathcal{C}(G/H_{n-1}, G/H_n)] \otimes T(G/H_n),$$

where the sum is taken over all distinct sequences $(H_0, \ldots, H_n)$ of subgroups in $\mathcal{F}$. The action of $\mathbf{Z}G$ is the natural left action, and we define $\mathbf{Z}G$-homomorphisms $d_n\colon W_n(T,\mathcal{F}) \to W_{n-1}(T,\mathcal{F})$ by $d_n(x) = \sum_{i=0}^n (-1)^i F_i(x)$, where F_i is given on generators by

$$F_i(h[f_1,\ldots,f_n]t) = \begin{cases} f_1(y)[f_2,\ldots,f_n]t & \text{if } i=0; \\ y[f_1,\ldots,f_{n-1}]f_n^*(t) & \text{if } i=n; \\ y[f_1,\ldots,f_if_{i+1},\ldots,f_n]t & \text{otherwise.} \end{cases}$$

That d_* is indeed a differential of this graded $\mathbf{Z}G$-module is easy to check.

Construction 2.4 may be thought of as a two-sided algebraic Bar construction $B_*(\mathcal{O}, \mathbf{Z}\mathcal{G}(\mathcal{F}), T)$ analogous to Elmendorf's geometric construction in [**E1**]. Here, $\mathcal{O}$ denotes the object space of $\mathbf{Z}\mathcal{G}(\mathcal{F})$, and the general construction is fully explained in [**E1**].

We consider now the basic properties of $W_*(T, \mathcal{F})$.

PROPOSITION 2.5. *Let $H \in \mathcal{F}$. Then $(W_*(T,\mathcal{F})^H, d_*^H)$ is a resolution of $T(G/H)$ by $\mathbf{Z}N(H)$-modules.*

PROOF. We have a natural isomorphism

$$\sigma\colon W_*(T,\mathcal{F})^H \cong \operatorname{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, W_*(T,\mathcal{F}))$$

of $\mathbf{Z}N(H)$-modules, specified by sending x to the unique $\mathbf{Z}G$-map $f\colon \mathbf{Z}G/H \to W_*(T,\mathcal{F})$ specified by $f(eH) = x$.

Further,

$$\begin{aligned} \operatorname{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, \mathbf{Z}G/K \otimes M) &\cong \operatorname{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, \mathbf{Z}G/K) \otimes M \\ &= \mathbf{Z}\mathcal{G}(\mathcal{F})(G/H, G/K) \otimes M \end{aligned}$$

for any trivial $\mathbf{Z}G$-module M. This gives

$$W_n(T,\mathcal{F})^H \cong \bigoplus \mathcal{C}(G/H, G/H_0) \otimes [\mathcal{C}(G/H_0, G/H_1) \otimes \cdots \otimes \mathcal{C}(G/H_{n-1}, G/H_n)] \otimes T(G/H_n),$$

with the notation of 2.4. Under this identification, define ϵ: $W_0(T,\mathcal{F})^H \to T(G/H)$ by taking a summand $f[\,]t$ to $f^*(t)$. A contracting chain homotopy is then given by

$$s_n(f[f_1,\ldots,f_n]t) = (1_H[f,f_1,\ldots,f_n]t)$$

where 1_H is the identity morphism $G/H \to G/H$. It is now formal that $ds - sd = 1$. □

REMARKS 2.6. (i) Note that $W_*(T,\mathcal{F})^H$ is not acyclic as a $\mathbf{Z}N(H)$-module; the contracting homotopy does not respect the $N(H)$-action.

(ii) If $\mathcal{F} = \{1\}$ and $T = \mathbf{Z}$, then $W_*(T,\mathcal{F})$ is a free $\mathbf{Z}G$-resolution of the trivial G-module $\mathbf{Z}$, and is, in fact, isomorphic with the usual Bar resolution.

(iii) $\mathbf{Z}G/H$ is not, in general, a projective $\mathbf{Z}G$-module. It is therefore inappropriate to view $W_*(T,\mathcal{F})$ as a resolution in the category of $\mathbf{Z}G$-modules.

Regard $W_*(T,\mathcal{F})$ as an $\mathcal{F}$-Hecke functor as in 2.3; we set $W_*(T,\mathcal{F})(G/H) = W_*(T,\mathcal{F})^H \cong \mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, W_*(T,\mathcal{F}))$. (Note that this isomorphism demonstrates the action of $\mathbf{Z}\mathcal{G}(\mathcal{F})$ as the obvious left one.)

PROPOSITION 2.7. *Let $H \in \mathcal{F}$. Then $\mathbf{Z}G/H$ is a projective $\mathcal{F}$-Hecke functor.*

PROOF. Denote the category of $\mathcal{F}$-Hecke functors by $\mathcal{M}(\mathcal{F})$. We now claim that $\mathcal{M}(\mathcal{F})(\mathbf{Z}G/H,T) \cong T(G/H)$, naturally in T, for every $T \in \mathcal{M}(\mathcal{F})$. Indeed, define ϕ: $\mathcal{M}(\mathcal{F})(\mathbf{Z}G/H,T) \to T(G/H)$ by taking f to $f(G/H)(eH)$ and ψ: $T(G/H) \to \mathcal{M}(\mathcal{F})(\mathbf{Z}G/H,T)$ by taking $\psi(t)(G/K)$ to be the composite

$$(\mathbf{Z}G/H)^K \overset{\sigma}{\cong} \mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/K, \mathbf{Z}G/H) \xrightarrow{\epsilon} T(G/K),$$

where $\epsilon(r) = T(r)(t)$. Thus $\psi(t)(G/K)(x) = T(\sigma(x))(t)$. That $\psi(t)$ is indeed a morphism in $\mathcal{M}(\mathcal{F})$ follows by the naturality of the definition of ψ, and that $\phi\psi = 1$ is clear. That $\psi\phi$ is the identity is a consequence of the following. Given any morphism f: $\mathbf{Z}G/H \to T$ in $\mathcal{M}(\mathcal{F})$ and $x \in (\mathbf{Z}G/H)^K$ with $K \in \mathcal{F}$, the diagram

$$\begin{tikzcd}
(\mathbf{Z}G/H)^H \arrow[r, "f(G/H)"] \arrow[d, "\sigma(x)^*"'] & T(G/H) \arrow[d, "T(\sigma(x))"] \\
(\mathbf{Z}G/H)^K \arrow[r, "f(G/K)"] & T(G/K)
\end{tikzcd}$$

must commute, where $\sigma(x)^*$ is induced by $\sigma(x)$: $\mathbf{Z}G/K \to \mathbf{Z}G/H$; $eK \mapsto x$. Chasing eH around the diagram gives

$$f(G/K)(\sigma(x)^* eH) = f(G/K)(x) = T(\sigma(x))(f(G/H)(eH)),$$

showing that $f(G/K)(x)$ is completely specified by $f(G/H)(eH)$ via application of $T(\sigma(x))$. This shows that $\psi\phi = 1$ as claimed.

For projectivity of $\mathbf{Z}G/H$ as an object in $\mathcal{M}(\mathcal{F})$, we now observe that one may complete any diagram of the form

$$\begin{tikzcd}
 & T' \arrow[d, "p"] \\
\mathbf{Z}G/H \arrow[ur, dashed, "g"] \arrow[r, "f"'] & T
\end{tikzcd}$$

in $\mathcal{M}(\mathcal{F})$ by setting $g = \psi(t)$ where $t \in T'(G/H)$ is any element in $p^{-1}(f(G/H)(eH))$. $\square$

REMARKS 2.8. If $H \notin \mathcal{F}$, then the assertion is false; let $\mathcal{F} = \{1\}$ and $H = G$. Then $\mathbf{Z}G/G = \mathbf{Z}$ is not a projective $\mathbf{Z}G$-module. Hence it is not projective in $\mathcal{M}(\mathcal{F})$, as one notes that if $\mathcal{F} = \{1\}$, $\mathcal{M}(\mathcal{F})$ is equivalent to the category of $\mathbf{Z}G$-modules. $\mathbf{Z}$ is, however, a projective $\mathcal{M}(\{1, g\})$-module by the proposition.

If each $T(G/H)$ is a projective $\mathbf{Z}$-module, then it follows that $W_*(T, \mathcal{F})$ is a projective resolution of T in the category $\mathcal{M}(\mathcal{F})$. (One should not expect the Bar resolution to yield projective modules in general—even in the case G trivial.) As far as the applications are concerned, we may take each $T(G/H)$ to be projective. In general, there are "enough" projectives; if $T \in \mathcal{M}(\mathcal{F})$, and if $F(G/H)$ is the free abelian group on the elements of $T(G/H)$, then the map

$$\epsilon\colon \bigoplus_{H \in \mathcal{F}} \mathbf{Z}G/H \otimes F(G/H) \to T$$

of $\mathcal{F}$-Mackey functors, defined just as in the proof of Proposition 2.5, is an epimorphism in $\mathcal{M}(\mathcal{F})$.

Now let M_* be any d.g. $\mathcal{F}$-Hecke functor, and let $T \in \mathcal{M}(\mathcal{F})$.

DEFINITION 2.9. The cohomology of M_* with T-coefficients, $H^*(M_*; T)$, is defined to be $H^*(\mathcal{M}(\mathcal{F})(M_*, T))$, (where, as usual, $\mathcal{M}(\mathcal{F})$(-,-) denotes Hom in the category $\mathcal{M}(\mathcal{F})$).

EXAMPLES 2.10. (i) Let X be any G-CW complex with cells of type $G/H \times D^n$ with $H \in \mathcal{F}$. Then the cellular chain complex, $C_*(X)$ is a d.g. $\mathcal{F}$-Hecke functor if we define $C_*(X)(G/H) = C_*(X)^H$ ($\neq C_*(X^H)$). Since $C_*(X)$ is a sum of G-modules $\mathbf{Z}G/H$ with $H \in \mathcal{F}$, it is a projective $\mathcal{F}$-Hecke functor by Proposition 2.7. We shall see in §3 that $H^*(C_*(X); T)$ is just Bredon cohomology, $H^*_G(X; T)$, as in [**B1**], provided $\mathcal{F}$ has the property that $H \in \mathcal{F}$, $K \subset H$ implies $K \in \mathcal{F}$. (This condition also allows one to think of T as a coefficient system defined on all orbit-types, so that H^*_G(-; T) is meaningful.)

(ii) If $M_* = W_*(T, \mathcal{F})$ with $\mathcal{F} = \{1\}$ and $T = \hat{\mathbf{Z}}$, it follows that $H^*(M_*; \hat{R}) \cong H^*(G; R)$ for all $\mathbf{Z}G$-modules R.

(iii) In general, define $H^*((T, \mathcal{F}); T')$ to be $H^*(W_*(T, \mathcal{F}); T')$. This generalizes the notion of $H^*(G; R)$ to larger families of subgroups, and is independent of the choice of resolution $W_*(T, \mathcal{F})$ if each $T(G/H)$ is projective.

3. Applications. Let $\mathcal{F}$ be a family of subgroups with the property that, if $H \in \mathcal{F}$ and $gKg^{-1} \subset H$ for some $g \in G$, then $K \in \mathcal{F}$.

By a universal G-space $E\mathcal{F}$, we shall mean a G-space with $E\mathcal{F}^H$ contractible for each $H \in \mathcal{F}$, and $E\mathcal{F}^H = \emptyset$ otherwise. (For example, $E\{1\} = EG$, the standard universal G-space.) These spaces were first constructed by Palais, while Elmendorf has an elegant construction in terms of the geometric two-sided Bar construction in [**E1**]. As a consequence of these constructions, we may assume that $E\mathcal{F}$ is a G-CW complex with cells of type $G/H \times D^n$ and $H \in \mathcal{F}$.

If T is in $\mathcal{M}(\mathcal{F})$, we consider the problem of computing $H^*_G(E\mathcal{F}; T)$, the Bredon cohomology of $E\mathcal{F}$. (As a special case, $H^*_G(EG; \hat{R}) \cong H^*(G; R)$.)

As promised in §2, we have the following result.

LEMMA 3.1. *Let X be any G-CW complex with cells of type $G/H \times D^n$ with $H \in \mathcal{F}$. Then $H^*(C_*(X); T) \cong H^*_G(X; T)$, Bredon cohomology of X with coefficients in the coefficient system T, where we take $T(G/K) = 0$ if $K \notin \mathcal{F}$.*

PROOF. It suffices to construct a natural transformation

$$\psi\colon H^*(C_*(X);T) \to H^*_G(X;T)$$

which restricts to an isomorphism where $X = G/H$ and $H \in \mathcal{F}$, and to verify that $H^*(C_*(X);T)$ is indeed a generalized equivariant cohomology theory in the sense of Bredon.

The latter claim is clear as a consequence of the projectivity of $C_*(X)$. For the former, recall from [**B1**] that $H^*_G(X;T)$ is the cohomology of $\mathrm{Hom}(\underline{C}_*(X),T)$ in the category of coefficient systems, where $\underline{C}_*(X)(G/H) = C_*(X^H)$ for all $H \subset G$. Since $X^H = \emptyset$ for $H \notin \mathcal{F}$, it follows that $\underline{C}_*(X)(G/H) = 0$ for such H. Now define $\psi\colon \mathcal{M}(\mathcal{F})(C_*(X),T) \to \mathrm{Hom}(\underline{C}_*(X),T)$ by

$$f(G/H) \mapsto \begin{cases} f(G/H)|C_*(X^H) & \text{if } H \in \mathcal{F}; \\ 0 & \text{if } H \notin \mathcal{F}. \end{cases}$$

Then ψ induces the desired transformation after passage to cohomology. Further, if $H \in \mathcal{F}$, then $\mathcal{M}(\mathcal{F})(C_*(G/H),T) \cong \mathcal{M}(\mathcal{F})(\mathbf{Z}G/H,T) \cong T(G/H)$, and under this isomorphism, ψ becomes the identity. □

PROPOSITION 3.2. *There exists an isomorphism, natural in T,*

$$\phi\colon H^*_G(E\mathcal{F};T) \cong H^*((\hat{\mathbf{Z}},\mathcal{F});T).$$

PROOF. By the lemmas,

$$H^*_G(E\mathcal{F};T) \cong H^*(C_*(E\mathcal{F});T), \text{ and } C_*(E\mathcal{F})^H \cong C_*(E\mathcal{F}/H)$$

for all $H \in \mathcal{F}$. Further, since $E\mathcal{F}^K$ is contractible for each $K \subset H$, it follows that $E\mathcal{F}$ is H-equivariantly contractible, whence $E\mathcal{F}/H$ is contractible. Thus $C_*(E\mathcal{F})$ is a projective resolution of $\hat{\mathbf{Z}}$ in $\mathcal{M}(\mathcal{F})$, and the result now follows by the lemma. □

REMARKS 3.3. Proposition 3.2 permits explicit computation of $H^*_G(E\mathcal{F};T)$ since one can write down the appropriate Bar resolution $W_*(T,\mathcal{F})$. It also follows that $H^*_G(E\mathcal{F};T)$ is an algebraic invariant of G, $\mathcal{F}$ and T. Of course, when T is replaced by $\hat{R}$ and $\mathcal{F}$ by $\{1\}$, we obtain the expected isomorphism $H^*_G(EG;\hat{R}) \cong H^*(G;R)$.

For the next application, we permit $\mathcal{F}$ to denote any family of subgroups closed under conjugation.

PROPOSITION 3.4. *Let X be a G-CW complex of dimension n such that X/H is $\mathbf{Z}$-acyclic for each $H \in \mathcal{F}$, and such that X has all orbits of the form G/H with $H \in \mathcal{F}$. Then $H^p((\hat{\mathbf{Z}},\mathcal{F});T) = 0$ for all $p > n$.*

PROOF. $C_*(X)$ is, under the hypothesis, a projective resolution of $\hat{\mathbf{Z}}$ in $\mathcal{M}(\mathcal{F})$, where $H^p(C_*(X);T) \cong H^p((\hat{\mathbf{Z}},\mathcal{F});T)$ for all p. Since the former group vanishes for $p > n$, the result follows. □

COROLLARY 3.5. *Let G act on a space X such that*

(i) *X inherits the structure of a contractible G-CW complex;*

(ii) $\dim X = n$;

(iii) *X has orbit-types G/H with $H \in \mathcal{F}$.*

Then $H^p((\hat{\mathbf{Z}},\mathcal{F});T) = 0$ for all $p > n$.

PROOF. By [**B2**, 5.4], $C_*(X/H)$ is $\mathbf{Z}$-acyclic for each $H \subset G$. The result now follows by 3.4. □

COROLLARY 3.6. *Let $G \in \mathcal{F}$. Then*

$$H^p((\hat{\mathbf{Z}}, \mathcal{F}); T) \cong \begin{cases} 0 & \text{if } p > 0, \\ T(G/G) & \text{if } p = 0 \end{cases}$$

for all T.

PROOF. We apply Corollary 3.5 with $X =$ point, obtaining $H^p((\hat{\mathbf{Z}}, \mathcal{F}); T) = 0$ for $p > 0$. The case $p = 0$ is then obtained by writing out $C*$(point) and observing that $\mathcal{M}(\mathcal{F})(\hat{\mathbf{Z}}, T) \cong T(G/G)$, as in the proof of 2.7. □

Corollary 3.5 gives the lowest possible dimension of a finite contractible CW-complex X with a suitable G-action. This problem has been examined by Oliver and Assadi (see [**A1**]), and they obtain necessary and sufficient conditions on the subgroups of G for such actions to exist.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903