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CYCLES OF PARTITIONS

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ABSTRACT. An operation on the set of partitions of a number n is introduced and the possible cycles are determined.

The problem to be discussed in the following has been circulating for some time. Let S be a natural number and let λ be a partition of S of length l with parts $\lambda_1, \lambda_2, \ldots, \lambda_l$. Define $T(\lambda)$ as the partition of S with parts $\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_l - 1$, l, ignoring any zeros that might occur. Assume that $S = 1 + 2 + \cdots + N$. Then it turns out that repeated application of T always yields the partition $1, 2, \ldots, N$. We shall prove that this is indeed so.

If S is arbitrary, repeated application of T leads into a cycle of partitions since there are only a finite number of these. We want to determine all cycles corresponding to S. Now, a cycle of partitions is completely determined by the sequence of the consecutive lengths of the partitions in the cycle. Due to this observation we define sets M_n by

$$M_n = \{\sigma = (\sigma_i)_{i \in Z} \mid \max \sigma_i = n, \forall i \colon \sigma_i = |\{\sigma_j \mid j < i, \sigma_j \geq i - j\}|\}.$$

A cycle of partitions then corresponds to a $\sigma \in M_n$, where n is the maximal length of a partition in the cycle. If we define σ, $\tau \in M_n$ to be equivalent if there exists an integer c such that $\sigma_{i+c} = \tau_i$ for all i, then the cycle determines a unique class in M_n. Conversely, a class in M_n uniquely determines a cycle of partitions by regarding the σ_i's as the lengths of the partitions in a cycle.

The above discussion shows that we may find all cycles of partitions by determining the set of equivalence classes in M_n and this is what we shall do. First, it is not hard to see that if $\sigma \in M_n$ then there exists a $p \in \mathbf{N}$ such that $\sigma_{i+p} = \sigma_i$ for all i. We call the smallest such p the period of σ and denote it by $p(\sigma)$.

LEMMA 1. *Let $\sigma \in M_n$. If $\sigma_i = n$ then $\forall k \in \mathbf{Z} \colon \sigma_{i+kn} = n$.*

PROOF. By definition of M_n, $\sigma_{i-n} = n$ since $n = \max \sigma_i$. Hence the lemma holds for negative k. Now $\sigma_{i+np(\sigma)} = n$, so by what we just observed $\sigma_{i+n} = n$ as well and we are done. Q.E.D.

PROPOSITION 2. *Let $\sigma \in M_n$. Then $\forall i \colon \sigma_i \in \{n, n-1\}$.*

PROOF. We may assume $n > 1$. Define $\sigma' = (\sigma'_i)_{i \in Z}$ by

$$\begin{aligned} \sigma'_i &= \sigma_i && \text{if } \sigma_i < n, \\ \sigma'_i &= n - 1 && \text{if } \sigma_i = n. \end{aligned}$$

Using Lemma 1 it follows that $\sigma' \in M_{n-1}$. We may assume $\sigma_0 = n$. Now, for all

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$i \in Z$ there exist integers x and y such that $i = xn + y(n-1)$. Then by Lemma 1 we obtain

$$\sigma'_{xn} = n-1 \Rightarrow \sigma'_i = \sigma'_{xn+y(n-1)} = n-1.$$

Hence $\sigma' = (n-1)_{i \in Z}$ which implies the result about σ. Q.E.D.

COROLLARY 3. $\sigma \in M_n \Rightarrow p(\sigma)|n$.

PROOF. Combine Lemma 1 with Proposition 2. Q.E.D.

Define for $\sigma \in M_n$, $S_\sigma = \sum_{i=1}^{n} \sigma_i - \binom{n}{2}$. If we consider the cycle of partitions corresponding to σ it is easy to see that the cycle consists of partitions of S_σ.

THEOREM 4. *Let $\sigma \in M_n$ and assume $S_\sigma = 1 + 2 + \cdots + N$. Then $N = n$ and M_n has just one class with sum S_σ, namely $\sigma = (n)_{i \in Z}$.*

PROOF. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ consist of a $n-1$'s and $n-a$ n's. Then $S_\sigma = \binom{n+1}{2} - a$, $0 \le a \le n-1$. Simple arithmetic now shows that $n = N$ and $a = 0$. Q.E.D.

Theorem 4 proves that if we start with any partition of $S = 1 + 2 + \cdots + N$ and apply T a number of times then we always end with the partition $1, 2, \ldots, N$ (corresponding to $\sigma = (n)_{i \in Z}$).

To determine all cycles for arbitrary S notice that there exist unique n and a such that $S = \binom{n+1}{2} - a$, $0 \le a \le n-1$. Let $C_a(n)$ denote the number of classes in M_n for which $\sigma_1, \ldots, \sigma_n$ contain a $n-1$'s and $n-a$ n's. Then $C_a(n)$ is the number of cycles of partitions of S.

THEOREM 5. $C_a(n) = \frac{1}{n} \sum_{d|(n,a)} \varphi(d) \binom{n/d}{a/d}$, *where φ is Euler's function.*

PROOF. If $\sigma_1, \ldots, \sigma_n$ is any sequence consisting of a $n-1$'s and $n-a$ n's, we define the sequence σ by periodically extending to both sides. It is clear that $\sigma \in M_n$. Thus $C_a(n)$ is equal to the number of circular words of length n with a letters $n-1$, and $n-a$ letters n. The standard Pólya enumeration theory (see e.g. [**1**]) applies. If one carries out the details one obtains the formula in the theorem. Q.E.D.

As an example of how to find all cycles corresponding to a specific S we consider $S = 8$. Here $n = 4$, $a = 2$, so there are $C_2(4) = 2$ cycles. The classes in M_4 corresponding to these cycles are represented by $\ldots, 4, 3, 4, 3, 4, 3, \ldots$ and $\ldots, 4, 4, 3, 3, 4, 4, \ldots$. Transforming this into cycles we get

$$\begin{tikzcd} & 2\ 2\ 4 \arrow[dr] & \\ 1\ 1 \arrow[ur] & & 3\ 3 \end{tikzcd} \qquad \begin{tikzcd} & 1\ 1\ 2\ 4 \arrow[dr] & \\ 1\ 2\ 2\ 3 \arrow[ur] & & 1\ 3\ 4 \arrow[dl] \\ & 2\ 3\ 3 \arrow[ul] & \end{tikzcd}$$

Finally we have some further comments on the special case $S = 1 + 2 + \cdots + N$. Theorem 4 shows that the partitions of S can be arranged in a tree so that the vertices correspond to the partitions and going down corresponds to applying T. The root of the tree is of course the partition $1, 2, \ldots, N$. For $S = 3, 6$ and 10 one may draw these trees by hand but this soon becomes impracticable since the number of partitions grows rapidly. For $S = 15$ there are 176 partitions. A

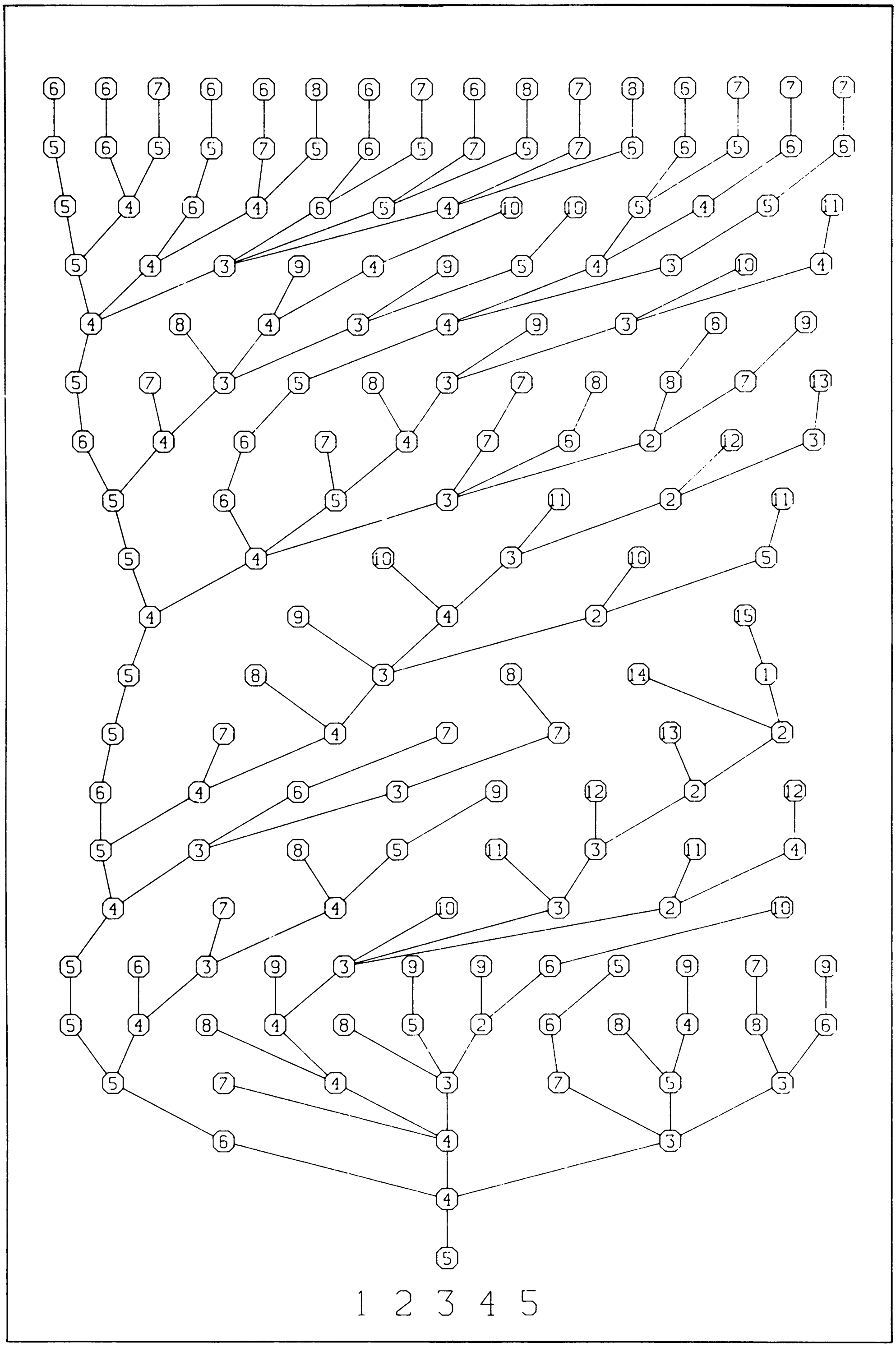

FIGURE 1

computer program was written in collaboration with C. B. Hansen to generate and draw these trees. We have included the tree for $S = 15$ as Figure 1. The vertices are labeled with the length of the partitions. Notice that there are $21 = 5^2 - 5 + 1$ levels in the tree. This appears to generalize to $N^2 - N + 1$ levels in the tree for $S = 1 + 2 + \cdots + N$.

REFERENCES

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