Cycles of Partitions

Jørgen Brandt

Abstract. An operation on the set of partitions of a number \( n \) is introduced and the possible cycles are determined.

The problem to be discussed in the following has been circulating for some time. Let \( S \) be a natural number and let \( \lambda \) be a partition of \( S \) of length \( l \) with parts \( \lambda_1, \lambda_2, \ldots, \lambda_l \). Define \( T(\lambda) \) as the partition of \( S \) with parts \( \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_l - 1, 1 \), ignoring any zeros that might occur. Assume that \( S = 1 + 2 + \cdots + N \). Then it turns out that repeated application of \( T \) always yields the partition \( 1, 2, \ldots, N \). We shall prove that this is indeed so.

If \( S \) is arbitrary, repeated application of \( T \) leads into a cycle of partitions since there are only a finite number of these. We want to determine all cycles corresponding to \( S \). Now, a cycle of partitions is completely determined by the sequence of the consecutive lengths of the partitions in the cycle. Due to this observation we define sets \( M_n \) by

\[
M_n = \{ \sigma = (\sigma_i)_{i \in \mathbb{Z}} | \max \sigma_i = n, \forall i: \sigma_i = |\{ \sigma_j | j < i, \sigma_j \geq i - j \}| \}.
\]

A cycle of partitions then corresponds to a \( \sigma \in M_n \), where \( n \) is the maximal length of a partition in the cycle. If we define \( \sigma, \tau \in M_n \) to be equivalent if there exists an integer \( c \) such that \( \sigma_{i+c} = \tau_i \) for all \( i \), then the cycle determines a unique class in \( M_n \). Conversely, a class in \( M_n \) uniquely determines a cycle of partitions by regarding the \( \sigma_i \)'s as the lengths of the partitions in a cycle.

The above discussion shows that we may find all cycles of partitions by determining the set of equivalence classes in \( M_n \) and this is what we shall do. First, it is not hard to see that if \( \sigma \in M_n \) then there exists a \( p \in \mathbb{N} \) such that \( \sigma_i + p = \sigma_i \) for all \( i \). We call the smallest such \( p \) the period of \( \sigma \) and denote it by \( p(\sigma) \).

Lemma 1. Let \( \sigma \in M_n \). If \( \sigma_i = n \) then \( \forall k \in \mathbb{Z}: \sigma_{i+kn} = n \).

Proof. By definition of \( M_n \), \( \sigma_{i-n} = n \) since \( n = \max \sigma_i \). Hence the lemma holds for negative \( k \). Now \( \sigma_{i+n} = n \), so by what we just observed \( \sigma_{i+n} = n \) as well and we are done. Q.E.D.

Proposition 2. Let \( \sigma \in M_n \). Then \( \forall i: \sigma_i \in \{ n, n - 1 \} \).

Proof. We may assume \( n > 1 \). Define \( \sigma' = (\sigma'_i)_{i \in \mathbb{Z}} \) by

\[
\sigma'_i = \begin{cases} 
\sigma_i & \text{if } \sigma_i < n, \\
n - 1 & \text{if } \sigma_i = n.
\end{cases}
\]

Using Lemma 1 it follows that \( \sigma' \in M_{n-1} \). We may assume \( \sigma_0 = n \). Now, for all
there exist integers $x$ and $y$ such that $i = xn + y(n - 1)$. Then by Lemma 1 we obtain

$$
s'_{xn} = n - 1 \Rightarrow s'_i = s'_{xn+y(n-1)} = n - 1.
$$

Hence $s' = (n - 1)i \in Z$ which implies the result about $\sigma$. Q.E.D.

**Corollary 3.** $\sigma \in M_n \Rightarrow p(\sigma)n$.

**Proof.** Combine Lemma 1 with Proposition 2. Q.E.D.

**Theorem 4.** Let $\sigma \in M_n$ and assume $S_\sigma = 1 + 2 + \cdots + N$. Then $N = n$ and $M_n$ has just one class with sum $S_\sigma$, namely $\sigma = (n)_{\in Z}$.

**Proof.** Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ consist of $n - 1$'s and $n - a$'s. Then $S_\sigma = \binom{n+1}{2} - a$, $0 \leq a \leq n - 1$. Simple arithmetic now shows that $n = N$ and $a = 0$. Q.E.D.

Theorem 4 proves that if we start with any partition of $S = 1 + 2 + \cdots + N$ and apply $T$ a number of times then we always end with the partition $1, 2, \ldots, N$ (corresponding to $\sigma = (n)_{\in Z}$).

To determine all cycles for arbitrary $N$ notice that there exist unique $n$ and $a$ such that $S = \binom{n+1}{2} - a$, $0 \leq a \leq n - 1$. Let $C_a(n)$ denote the number of classes in $M_n$ for which $\sigma_1, \ldots, \sigma_n$ contain a $n - 1$'s and $n - a$'s. Then $C_a(n)$ is the number of cycles of partitions of $S$.

**Theorem 5.** $C_a(n) = \frac{1}{n} \sum_{d \mid (n,a)} \varphi(d) \binom{n/d}{a/d}$, where $\varphi$ is Euler's function.

**Proof.** If $\sigma_1, \ldots, \sigma_n$ is any sequence consisting of $n - 1$'s and $n - a$'s, we define the sequence $\sigma$ by periodically extending to both sides. It is clear that $\sigma \in M_n$. Thus $C_a(n)$ is equal to the number of circular words of length $n$ with $a$ letters $n - 1$, and $n - a$ letters $n$. The standard Polya enumeration theory (see e.g. [1]) applies. If one carries out the details one obtains the formula in the theorem. Q.E.D.

As an example of how to find all cycles corresponding to a specific $S$ we consider $S = 8$. Here $n = 4$, $a = 2$, so there are $C_2(4) = 2$ cycles. The classes in $M_4$ corresponding to these cycles are represented by $\ldots, 4, 3, 3, 4, 3, \ldots$ and $\ldots, 4, 4, 3, 3, 4, 4, \ldots$. Transforming this into cycles we get

$$
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
2 & 2 & 4 & \ \ \\
1 & 1 & 3 & 3 \\
\end{array}
$$

Finally we have some further comments on the special case $S = 1 + 2 + \cdots + N$. Theorem 4 shows that the partitions of $S$ can be arranged in a tree so that the vertices correspond to the partitions and going down corresponds to applying $T$. The root of the tree is of course the partition $1, 2, \ldots, N$. For $S = 3, 6$ and $10$ one may draw these trees by hand but this soon becomes impracticable since the number of partitions grows rapidly. For $S = 15$ there are 176 partitions. A
FIGURE 1
computer program was written in collaboration with C. B. Hansen to generate and draw these trees. We have included the tree for \( S = 15 \) as Figure 1. The vertices are labeled with the length of the partitions. Notice that there are \( 21 = 5^2 - 5 + 1 \) levels in the tree. This appears to generalize to \( N^2 - N + 1 \) levels in the tree for \( S = 1 + 2 + \cdots + N \).

REFERENCES


MATEMATISK INSTITUT, AARHUS UNIVERSITET, 8000 AARHUS, DK DANMARK