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# CYCLES OF PARTITIONS

JØRGEN BRANDT

ABSTRACT. An operation on the set of partitions of a number $n$ is introduced and the possible cycles are determined.

The problem to be discussed in the following has been circulating for some time. Let $S$ be a natural number and let $\lambda$ be a partition of $S$ of length $l$ with parts $\lambda_1, \lambda_2, \ldots, \lambda_l$. Define $T(\lambda)$ as the partition of $S$ with parts $\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_l - 1, l$, ignoring any zeros that might occur. Assume that $S = 1 + 2 + \cdots + N$. Then it turns out that repeated application of $T$ always yields the partition $1, 2, \ldots, N$. We shall prove that this is indeed so.

If $S$ is arbitrary, repeated application of $T$ leads into a cycle of partitions since there are only a finite number of these. We want to determine all cycles corresponding to $S$. Now, a cycle of partitions is completely determined by the sequence of the consecutive lengths of the partitions in the cycle. Due to this observation we define sets $M_n$ by

$$M_n = \{\sigma = (\sigma_i)_{i \in Z} \mid \max \sigma_i = n, \forall i\colon \sigma_i = |\{\sigma_j | j < i, \sigma_j \geq i - j\}|\}.$$

A cycle of partitions then corresponds to a $\sigma \in M_n$, where $n$ is the maximal length of a partition in the cycle. If we define $\sigma$, $\tau \in M_n$ to be equivalent if there exists an integer $c$ such that $\sigma_{i+c} = \tau_i$ for all $i$, then the cycle determines a unique class in $M_n$. Conversely, a class in $M_n$ uniquely determines a cycle of partitions by regarding the $\sigma_i$'s as the lengths of the partitions in a cycle.

The above discussion shows that we may find all cycles of partitions by determining the set of equivalence classes in $M_n$ and this is what we shall do. First, it is not hard to see that if $\sigma \in M_n$ then there exists a $p \in \mathbf{N}$ such that $\sigma_{i+p} = \sigma_i$ for all $i$. We call the smallest such $p$ the period of $\sigma$ and denote it by $p(\sigma)$.

LEMMA 1. *Let $\sigma \in M_n$. If $\sigma_i = n$ then $\forall k \in \mathbf{Z}\colon \sigma_{i+kn} = n$.*

PROOF. By definition of $M_n$, $\sigma_{i-n} = n$ since $n = \max \sigma_i$. Hence the lemma holds for negative $k$. Now $\sigma_{i+np(\sigma)} = n$, so by what we just observed $\sigma_{i+n} = n$ as well and we are done. Q.E.D.

PROPOSITION 2. *Let $\sigma \in M_n$. Then $\forall i\colon \sigma_i \in \{n, n-1\}$.*

PROOF. We may assume $n > 1$. Define $\sigma' = (\sigma'_i)_{i \in Z}$ by

$$\begin{aligned} \sigma'_i &= \sigma_i && \text{if } \sigma_i < n, \\ \sigma'_i &= n - 1 && \text{if } \sigma_i = n. \end{aligned}$$

Using Lemma 1 it follows that $\sigma' \in M_{n-1}$. We may assume $\sigma_0 = n$. Now, for all

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$i \in Z$ there exist integers $x$ and $y$ such that $i = xn + y(n-1)$. Then by Lemma 1 we obtain

$$\sigma'_{xn} = n - 1 \Rightarrow \sigma'_i = \sigma'_{xn+y(n-1)} = n - 1.$$

Hence $\sigma' = (n-1)_{i \in Z}$ which implies the result about $\sigma$. Q.E.D.

COROLLARY 3. $\sigma \in M_n \Rightarrow p(\sigma) | n$.

PROOF. Combine Lemma 1 with Proposition 2. Q.E.D.

Define for $\sigma \in M_n$, $S_\sigma = \sum_{i=1}^n \sigma_i - \binom{n}{2}$. If we consider the cycle of partitions corresponding to $\sigma$ it is easy to see that the cycle consists of partitions of $S_\sigma$.

THEOREM 4. *Let $\sigma \in M_n$ and assume $S_\sigma = 1 + 2 + \cdots + N$. Then $N = n$ and $M_n$ has just one class with sum $S_\sigma$, namely $\sigma = (n)_{i \in Z}$.*

PROOF. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ consist of $a$ $n-1$'s and $n-a$ $n$'s. Then $S_\sigma = \binom{n+1}{2} - a$, $0 \le a \le n-1$. Simple arithmetic now shows that $n = N$ and $a = 0$. Q.E.D.

Theorem 4 proves that if we start with any partition of $S = 1 + 2 + \cdots + N$ and apply $T$ a number of times then we always end with the partition $1, 2, \ldots, N$ (corresponding to $\sigma = (n)_{i \in Z}$).

To determine all cycles for arbitrary $S$ notice that there exist unique $n$ and $a$ such that $S = \binom{n+1}{2} - a$, $0 \le a \le n-1$. Let $C_a(n)$ denote the number of classes in $M_n$ for which $\sigma_1, \ldots, \sigma_n$ contain $a$ $n-1$'s and $n-a$ $n$'s. Then $C_a(n)$ is the number of cycles of partitions of $S$.

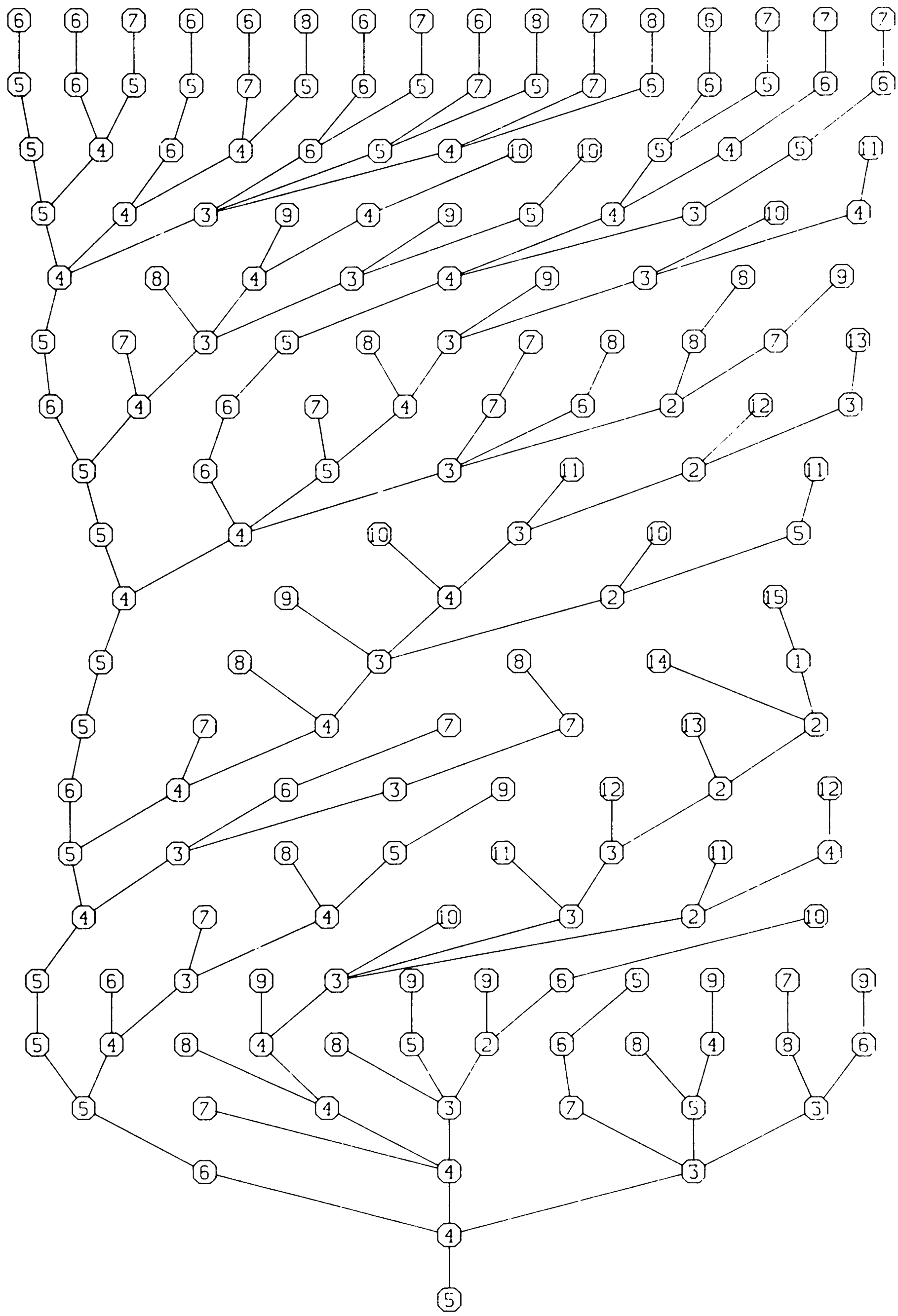
THEOREM 5. $C_a(n) = \frac{1}{n} \sum_{d|(n,a)} \varphi(d) \binom{n/d}{a/d}$, *where $\varphi$ is Euler's function.*

PROOF. If $\sigma_1, \ldots, \sigma_n$ is any sequence consisting of $a$ $n-1$'s and $n-a$ $n$'s, we define the sequence $\sigma$ by periodically extending to both sides. It is clear that $\sigma \in M_n$. Thus $C_a(n)$ is equal to the number of circular words of length $n$ with $a$ letters $n-1$, and $n-a$ letters $n$. The standard Pólya enumeration theory (see e.g. [**1**]) applies. If one carries out the details one obtains the formula in the theorem. Q.E.D.

As an example of how to find all cycles corresponding to a specific $S$ we consider $S = 8$. Here $n = 4$, $a = 2$, so there are $C_2(4) = 2$ cycles. The classes in $M_4$ corresponding to these cycles are represented by $\ldots, 4, 3, 4, 3, 4, 3, \ldots$ and $\ldots, 4, 4, 3, 3, 4, 4, \ldots$. Transforming this into cycles we get

$$\begin{tikzcd} 2\,2\,4 \arrow[d, shift left] \\ 1\,1\;\;3\,3 \arrow[u, shift left] \end{tikzcd} \qquad \begin{tikzcd} & 1\,1\,2\,4 \arrow[dr] & \\ 1\,2\,2\,3 \arrow[ur] & & 1\,3\,4 \arrow[dl] \\ & 2\,3\,3 \arrow[ul] & \end{tikzcd}$$

Finally we have some further comments on the special case $S = 1 + 2 + \cdots + N$. Theorem 4 shows that the partitions of $S$ can be arranged in a tree so that the vertices correspond to the partitions and going down corresponds to applying $T$. The root of the tree is of course the partition $1, 2, \ldots, N$. For $S = 3, 6$ and $10$ one may draw these trees by hand but this soon becomes impracticable since the number of partitions grows rapidly. For $S = 15$ there are 176 partitions. A



1 2 3 4 5

**FIGURE 1**

computer program was written in collaboration with C. B. Hansen to generate and draw these trees. We have included the tree for $S = 15$ as Figure 1. The vertices are labeled with the length of the partitions. Notice that there are $21 = 5^2 - 5 + 1$ levels in the tree. This appears to generalize to $N^2 - N + 1$ levels in the tree for $S = 1 + 2 + \cdots + N$.

## References

1. L. L. Dornhoff and F. E. Holm, *Applied modern algebra*, Macmillan, New York, 1978.

MATEMATISK INSTITUT, AARHUS UNIVERSITET, 8000 AARHUS, DK DANMARK