SHORTER NOTES

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A CLOSED INFINITE DIMENSIONAL REFLEXIVE LEFT INVARIANT SUBSPACE OF $L_\infty(G)$

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ABSTRACT. We give an example of a locally compact group $G$, for which $L_\infty(G)$ has a closed infinite dimensional reflexive left invariant subspace.

In a recent paper I. Glicksberg [2] showed that if $G$ is a compact or abelian group then every closed left invariant reflexive subspace of $L_\infty(G)$ must be finite dimensional. The purpose of this note is to exhibit a nonabelian, noncompact group for which $L_\infty(G)$ has an infinite dimensional closed left translation invariant reflexive subspace.

Let $H$ be an infinite dimensional Hilbert space and $B(H)$ the bounded operators on $H$. Let $G$ be the group of unitary operators in $B(H)$ with the discrete topology. Then the map $\pi: G \to B(H)$ is a continuous unitary representation of $G$ on $H$. We can represent $l_1(G)$ on $H$ in the usual way by defining

$$\pi(f)\xi = \sum_{x \in G} f(x)\pi(x)\xi \quad \text{for all} \quad \xi \in H.$$ 

Now fix $\xi \in H$, $\xi \neq 0$. We claim that $\pi(l_1(G))\xi = H$. For let $\eta \in H$, $\eta \neq 0$. Then there exists $x_0 \in G$ such that $\pi(x_0)\xi/\||\xi|| = \eta/\||\eta||$. Define $f$ on $G$ by $f(x_0) = ||\eta||/||\xi||$ and $f(x) = 0$ if $x \neq x_0$. Then $f \in l_1(G)$ and

$$\pi(f)\xi = \sum_{x \in G} f(x)\pi(x)\xi = f(x_0)\pi(x_0)\xi = \eta.$$ 

Since $\pi$ is a continuous representation of $l_1(G)$ on $H$, the map $f \to \pi(f)\xi$ is a continuous map of $l_1(G)$ onto $H$. Let $M = \{f \in l_1(G): \pi(f)\xi = 0\}$. Then $M$ is closed. Also since $\pi(xf) = \pi(x^{-1})\pi(f)$, $M$ is closed under left translation.

Let $\Phi: l_1(G)/M \to H$ be the quotient map. Then $\Phi$ is a one-to-one continuous surjection and so by the open mapping theorem it is bicontinuous. It follows that
the adjoint map $\Phi^*: H^* \to (l_1(G)/M)^*$ is one-to-one and bicontinuous. By [1, II4.28b] $(l_1(G)/M)^*$ is isometrically isomorphic with

$$M^\perp = \left\{ h \in l_\infty(G) : \sum_{x \in G} f(x)h(x) = 0 \text{ for all } f \in M \right\}.$$ 

We show that $M^\perp$ is a closed left invariant infinite dimensional reflexive subspace of $l_\infty(G)$. That $M^\perp$ is closed follows from the fact that it is the annihilator of $M$. The left invariance of $M^\perp$ is a consequence of the left invariance of $M$. For if $h \in M^\perp$, $x_0 \in G$ and $f \in M$ then

$$\sum_{x \in G} x_0h(x)f(x) = \sum_{x \in G} h(x_0x)f(x) = \sum_{x \in G} h(x)f(x_0^{-1}x) = \sum_{x \in G} h(x)x_0^{-1}f(x) = 0.$$ 

Finally, since $M^\perp$ is bicontinuously isomorphic with $H^*$, we have that $M^\perp$ is infinite dimensional and reflexive.

REFERENCES

2. I. Glicksberg, Reflexive invariant subspaces of $L_\infty(G)$ are finite dimensional, Math. Scand. 47 (1980), 308–310.

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