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SHORTER NOTES

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A CLOSED INFINITE DIMENSIONAL REFLEXIVE LEFT INVARIANT SUBSPACE OF $L_\infty(G)$

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ABSTRACT. We give an example of a locally compact group G, for which $L_\infty(G)$ has a closed infinite dimensional reflexive left invariant subspace.

In a recent paper I. Glicksberg [**2**] showed that if G is a compact or abelian group then every closed left invariant reflexive subspace of $L_\infty(G)$ must be finite dimensional. The purpose of this note is to exhibit a nonabelian, noncompact group for which $L_\infty(G)$ has an infinite dimensional closed left translation invariant reflexive subspace.

Let H be an infinite dimensional Hilbert space and $B(H)$ the bounded operators on H. Let G be the group of unitary operators in $B(H)$ with the discrete topology. Then the map $\pi\colon G \to B(H)$ is a continuous unitary representation of G on H. We can represent $l_1(G)$ on H in the usual way by defining

$$\pi(f)\xi = \sum_{x\in G} f(x)\pi(x)\xi \quad \text{for all } \xi \in H.$$

Now fix $\xi \in H$, $\xi \neq 0$. We claim that $\pi(l_1(G))\xi = H$. For let $\eta \in H$, $n \neq 0$. Then there exists $x_0 \in G$ such that $\pi(x_0)\xi/||\xi|| = \eta/||\eta||$. Define f on G by $f(x_0) = ||\eta||/||\xi||$ and $f(x) = 0$ if $x \neq x_0$. Then $f \in l_1(G)$ and

$$\pi(f)\xi = \sum_{x\in G} f(x)\pi(x)\xi = f(x_0)\pi(x_0)\xi = \eta.$$

Since π is a continuous representation of $l_1(G)$ on H, the map $f \to \pi(f)\xi$ is a continuous map of $l_1(G)$ onto H. Let $M = \{f \in l_1(G)\colon \pi(f)\xi = 0\}$. Then M is closed. Also since $\pi({}_xf) = \pi(x^{-1})\pi(f)$, M is closed under left translation.

Let $\Phi\colon l_1(G)/M \to H$ be the quotient map. Then Φ is a one-to-one continuous surjection and so by the open mapping theorem it is bicontinuous. It follows that

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the adjoint map $\Phi^*\colon H^* \to (l_1(G)/M)^*$ is one-to-one and bicontinuous. By [**1**, II4.28b] $(l_1(G)/M)^*$ is isometrically isomorphic with

$$M^{\perp} = \left\{ h \in l_\infty(G)\colon \sum_{x\in G} f(x)h(x) = 0 \text{ for all } f \in M \right\}.$$

We show that $M^{\perp}$ is a closed left invariant infinite dimensional reflexive subspace of $l_\infty(G)$. That $M^{\perp}$ is closed follows from the fact that it is the annihilator of M. The left invariance of $M^{\perp}$ is a consequence of the left invariance of M. For if $h \in M^{\perp}$, $x_0 \in G$ and $f \in M$ then

$$\sum_{x\in G} {}_{x_0}h(x)f(x) = \sum_{x\in G} h(x_0x)f(x) = \sum_{x\in G} h(x)f(x_0^{-1}x) = \sum_{x\in G} h(x)_{x_0^{-1}}f(x) = 0.$$

Finally, since $M^{\perp}$ is bicontinuously isomorphic with H^*, we have that $M^{\perp}$ is infinite dimensional and reflexive.

References

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