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A NOTE ON $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

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ABSTRACT. Denoting by $S(N)$ the number of natural numbers n less than N for which

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

has no solutions in positive integers, we show that $S(N) \ll N/\log^2 N$.

P. Erdös conjectured that the equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{1}$$

has positive integer solutions x, y, z for every natural number $n \geqslant 2$. This problem has attracted the attention of Straus, Bernstein, Shapiro, Oblath and Yamamoto (see [1] for precise references); Chao Ko, Chi Sun and S. J. Chang [2]; and R. W. Jollensten [3]. The best result hitherto obtained is that (1) holds for $n < 1.1 \times 10^7$.

In [4], W. A. Webb proved that (1) holds for almost all natural numbers. More precisely, let $S(N)$ denote the number of n's $(n < N)$ for which (1) has no solution. Then

$$S(N) \ll \frac{N}{\log^{7/4} N}.$$

Webb's proof is based on Selbeg's sieve method, and is quite complicated. Webb remarked that his technique could be used to improve the exponent from $7/4$ to 2. It is the aim of this note to give a simple proof of this slightly sharper result.

THEOREM A. $S(N) \ll N/\log^2 N$.

The proof is based on the following theorem [5, p. 70].

THEOREM B. *Let g be a natural number. Let a_i and b_i $(i = 1,2,\ldots,g)$ be pairs of integers satisfying $(a_i, b_i) = 1$ $(i = 1,2,\ldots,g)$ and define*

$$E = \prod_{i=1}^{g} a_i \prod_{1 \leqslant r < s \leqslant g} (a_r b_s - a_s b_r) \neq 0.$$

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Let y and x be real numbers satisfying $1 \leqslant y \leqslant x$. Further, let $\mathfrak{P}$ be a set of primes for which there exist constants δ and A such that

$$\sum_{\substack{p<y \\ p\in\mathfrak{P}}} \frac{1}{p} \geqslant \delta \log\log y - A.$$

Then

$$(2)\qquad \begin{aligned} &|\{n: x-y<n\leqslant x, ((a_i n+b_i), \mathfrak{P}) = 1 \text{ for } i=1,2,\dots,g\}| \\ &\qquad \ll \prod_{\substack{p|E \\ p\in\mathfrak{P}}} \left(1-\frac{1}{p}\right)^{\rho(p)-g} \frac{y}{\log^{\delta g} y} \end{aligned}$$

where $\rho(p)$ denotes the number of solutions of

$$\sum_{i=1}^{g} (a_i n + b_i) \equiv 0 \pmod p$$

and where the constant implied by the $\ll$ notation depends on g and A only.

PROOF OF THEOREM A. Obviously,

$$\frac{4}{n} = \begin{cases} \dfrac{1}{n(k+1)k} + \dfrac{1}{n(k+1)} + \dfrac{1}{vk}, & n = (4k-1)v, \\ \dfrac{1}{nk} + \dfrac{1}{nkv} + \dfrac{1}{vk}, & n+1 = (4k-1)v, \\ \dfrac{1}{nk} + \dfrac{1}{nk(kv-1)} + \dfrac{1}{kv-1}, & n+4 = (4k-1)v, \\ \dfrac{1}{nk} + \dfrac{1}{k(kv-n)} + \dfrac{1}{n(kv-n)}, & 4n+1 = (4k-1)v. \end{cases}$$

Hence, if one of the four numbers n, $n+1$, $n+4$, $4n+1$, has a factor of the form $4k-1$, then (1) holds.

Thus, we may choose

$$\mathfrak{P} = \{p: p \equiv -1 \pmod 4\}, \qquad y = x, g = 4,$$

and

$$\prod_{i=1}^{4} (a_i x + b_i) = x(x+1)(x+4)(4x+1)$$

in Theorem B and obtain,

$$E = 2^4 \cdot 3^3 \cdot 5 \neq 0, \qquad \rho(3) = 2,$$

and

$$\prod_{\substack{p|E \\ p\in\mathfrak{P}}} \left(1-\frac{1}{p}\right)^{\rho(p)-g} = \left(\frac{2}{3}\right)^{-2}.$$

By Mertens' result [**5**, p. 35],

$$\sum_{\substack{p<x \\ p\equiv l(\text{mod } k)}} \frac{1}{p} = \frac{1}{\varphi(k)} \log\log x + O_k(1), \qquad (l, k) = 1.$$

Taking $\delta = \frac{1}{2}$, we obtain Theorem A directly from (2).

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