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ON U_m-NUMBERS

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ABSTRACT. In this paper we shall give some examples of U_m-numbers by using the continued fraction expansions of algebraic numbers of degree $m > 1$.

DEFINITION.[1] Let ξ be a complex number and m $(m > 1)$ a positive integer. The number ξ is called a U_m-number if for every $w > 0$ there are infinitely many algebraic numbers γ of degree m with

$$0 < |\xi - \gamma| \leq H(\gamma)^{-w}$$

and if there exist constants $C > 0$ and K depending only on ξ and m such that the relation

$$|\xi - \beta| > CH(\beta)^{-K}$$

holds for every algebraic number β of degree $< m$. ($H(\gamma)$ is the maximum of the absolute value of coefficients of the minimal polynomial of γ [**1, 2, 7, 8**].)

THEOREM. *Let α $(\alpha > 1)$ be a real algebraic number of degree m $(m > 1)$ with continued fraction expansion*

$$\alpha = \langle a_0, a_1, a_2, \ldots, a_n, \ldots \rangle \tag{1}$$

and p_n/q_n $(n = 0, 1, \ldots)$ be nth convergent of the continued fraction (1). *Let $\{r_j\}$ and $\{s_j\}$ $(j = 0, 1, \ldots)$ be two sequences of nonnegative integers with the following properties*

$$0 = r_0 < s_0 < r_1 < s_1 < r_2 < s_2 < r_3 < s_3 < \cdots \qquad (r_{n+1} - s_n \geq 2), \tag{2}$$

$$\text{(a)} \lim_{n\to\infty} (\log q_{s_n}/\log q_{r_n}) = \infty, \qquad \text{(b)} \varlimsup_{n\to\infty} (\log q_{r_{n+1}}/\log q_{s_n}) < \infty. \tag{3}$$

Finally we define positive integers b_j $(j = 0, 1, 2, \ldots)$ by

$$b_j = \begin{cases} a_j & \text{if } r_n \leq j \leq s_n \ (n = 0, 1, \ldots) \\ \nu_j & \left(1 \leq \nu_j \leq K_1 q_j^{K_2}, \ \sum_{j=s_n+1}^{r_{n+1}-1} (a_j - \nu_j)^2 \neq 0\right) \\ & \text{if } s_n < j < r_{n+1} \ (n = 0, 1, \ldots) \end{cases} \tag{4}$$

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[1] We note that we have, in fact, defined a Koksma U_m^*-number instead of a Mahler U_m^*-number. However, it is known that they are the same (see [**6, 10**]).

where K_1, K_2 are positive integers. Then the real number ξ with continued fraction expansion

$$\xi = \langle b_0, b_1, \ldots, b_n, \ldots \rangle$$

is a U_m-number.

In the proof, we shall use some lemmas as follows.

LEMMA I. *Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ $(k \geq 1)$ be algebraic numbers belonging to an algebraic number field K of degree g, η be algebraic and $F(y, x_1, x_2, \ldots, x_k)$ be a polynomial with integral coefficients so that its degree is at least one in y. Next assume that $P(\eta, \alpha_1, \alpha_2, \ldots, \alpha_k) = 0$. Then degree of $\eta \leq d \cdot g$ and*

$$h_\eta \leq 3^{2dg+(l_1+l_2+\cdots+l_k)g} H^g h_{\alpha_1}^{l_1 g} h_{\alpha_2}^{l_2 g} \cdots h_{\alpha_k}^{l_k g},$$

where h_η is the height of η, h_{α_i} $(i = 1, 2, \ldots, k)$ is the height of α_i $(i = 1, 2, \ldots, k)$, H is the maximum of absolute value of coefficients of F, l_i $(i = 1, 2, \ldots, k)$ is the degree of F in x_i $(i = 1, 2, \ldots, k)$, and d is the degree of F in y (see O. S. Icen [5]).

LEMMA II. *Let α_1 and α_2 be two algebraic numbers such that they have different minimal polynomials. Let n_1 and n_2 be degrees of α_1, α_2 and $H(\alpha_1)$, $H(\alpha_2)$ be the height of α_1, α_2 respectively. Then we have*

$$|\alpha_1 - \alpha_2| \geq \left(2^{\max(n_1, n_2)-1}[(n_1+1)H(\alpha_1)]^{n_2}[(n_2+1)H(\alpha_2)]^{n_1}\right)^{-1} \tag{5}$$

(see R. Güting [4]).

In the following we will use certain elementary facts about continued fractions which the reader may find in Cassels [3].

LEMMA III. *Let P/Q $(P/Q > 1)$ be a rational integer with finite continued fraction*

$$\frac{P}{Q} = \langle a_0, a_1, a_2, \ldots, a_n, b_{n+1}, \ldots, b_m \rangle \tag{6}$$

and A_j/B_j $(j = 0, 1, 2, \ldots, n)$ be jth convergent of (6). *Put $R_n/S_n = \langle b_{n+1}, \ldots, b_m \rangle$. Then we have*

$$P = A_n R_n + A_{n-1} S_n, \qquad Q = B_n R_n + B_{n-1} S_n. \tag{7}$$

PROOF. We have from the theory of continued fractions that

$$\frac{P}{Q} = \left\langle a_0, a_1, \ldots, a_n, \frac{R_n}{S_n} \right\rangle = \frac{A_n R_n + A_{n-1} S_n}{B_n R_n + B_{n-1} S_n}.$$

Put $A_n R_n + A_{n-1} S_n = C$, $B_n R_n + B_{n-1} S_n = D$. Assume that $(C, D) = t$. Then we have

$$t \mid CB_n - A_n D = S_n, \qquad t \mid CB_{n-1} - A_{n-1} D = R_n$$

so we get

$$t \mid (S_n, R_n) = 1 \quad \text{or} \quad t = 1.$$

LEMMA IV. *Let P_1/Q_1, P_2/Q_2 be two rational numbers with continued fraction expansion*

$$\frac{P_1}{Q_1} = \langle a_0, a_1, \ldots, a_n \rangle, \qquad \frac{P_2}{Q_2} = \langle b_0, b_1, \ldots, b_n \rangle \qquad (a_0 > 0, b_0 > 0)$$

such that

$$b_j \leq S_1 a_j^{S_2} \qquad (j = 0, 1, 2, \ldots, n) \tag{8}$$

where S_1, S_2 are positive integers. Then we have

$$\max(P_2, Q_2) \leq a_0^{S_2} 2^{4(1+\log_2 S_1)} \max(P_1, Q_1)^{S_2+2(1+\log_2 S_1)}. \tag{9}$$

PROOF. We know from the theory of continued fractions that

$$Q_2 \leq 2^n \prod_{j=0}^{n} b_j, \qquad P_2 \leq 2 b_0 Q_2. \tag{10}$$

By using (8) in (10) we get

$$\max(P_2, Q_2) \leq 2^{n+1} b_0 \prod_{j=0}^{n} b_j \leq 2^{n+1} S_1^{n+2} a_0^{S_2} \left(\prod_{j=0}^{n} a_j \right)^{S_2}. \tag{11}$$

On the other hand we have

$$Q_1 \geq \max\left(\prod_{j=0}^{n} a_j, 2^{(n-2)/2} \right). \tag{12}$$

Thus, by combining the relations (11) and (12) we obtain

$$\max(P_2, Q_2) \leq a_0^{S_2} 2^{4(1+\log_2 S_1)} \max(P_1, Q_1)^{S_2+2(1+\log_2 S_1)}.$$

PROOF OF THE THEOREM. We define algebraic numbers α_{r_n} $(n = 0, 1, 2, \ldots)$ by

$$\alpha_{r_n} = \langle c_0, c_1, \ldots, c_n, \ldots \rangle, \tag{13}$$

where

$$c_r = \begin{cases} b_r, & r \leq r_n \\ a_r, & r > r_n \end{cases} \qquad (n = 0, 1, 2, \ldots).$$

Put

$$\beta_{r_n} = \langle a_{r_n+1}, a_{r_n+2}, \ldots \rangle \qquad (n = 0, 1, 2, \ldots), \tag{14}$$

$$\frac{p_k'}{q_k'} = \langle b_0, b_1, \ldots, b_k \rangle. \tag{15}$$

We see from the definitions of algebraic number α and β_{r_n} $(n = 0, 1, \ldots)$ that

$$\alpha = \langle a_0, a_1, \ldots, a_n, \beta_{r_n} \rangle$$

or

$$\alpha q_{r_n} \beta_{r_n} + q_{r_n-1} \alpha - \beta_{r_n} p_{r_n} - p_{r_n-1} = 0 \qquad (n = 0, 1, \ldots). \tag{16}$$

Now we can apply Lemma I with

$$F(y, x_1) = q_{r_n} y x_1 + q_{r_n - 1} x_1 - p_{r_n} y - p_{r_n - 1}, \qquad \eta = \beta_{r_n}, \alpha_1 = \alpha$$

and we get

$$H(\beta_{r_n}) \leqslant 3^{3m} H(\alpha)^m \max(p_{r_n}, q_{r_n})^m \qquad (n = 0, 1, \ldots),$$

or using the relation $p_{r_n} < 2a_0 q_{r_n}$ and putting $c_1 = 3^{3m}(2a_0)^m H(\alpha)^m$, we obtain

$$H(\beta_{r_n}) \leqslant c_1 q_{r_n}^m \qquad (n = 0, 1, \ldots). \tag{17}$$

Similarly combining the relations (13), (14), (15) we get

$$q'_{r_n} \beta_{r_n} \alpha_{r_n} + q'_{r_n - 1} \alpha_{r_n} - p'_{r_n} \beta_{r_n} - p'_{r_n - 1} = 0, \tag{18}$$

and applying Lemma I with $\eta = \alpha_{r_n}$, $\alpha_1 = \beta_{r_n}$ $(n = 0, 1, \ldots)$ and using (17)

$$H(\alpha_{r_n}) \leqslant 3^{3m} \left[\max(p'_{r_n}, q'_{r_n})\right]^m (c_1 q_{r_n}^m)^m \qquad (n = 0, 1, \ldots). \tag{19}$$

On the other hand, by (4), we have

$$b_j \leqslant K_1 a_j^{K_2} \qquad (j = 0, 1, \ldots).$$

Therefore we can apply Lemma IV with $S_1 = K_1$, $S_2 = K_2$, $P_1/Q_1 = p_{r_n}/q_{r_n}$, $P_2/Q_2 = p'_{r_n}/q'_{r_n}$ and we obtain

$$\max(p'_{r_n}, q'_{r_n}) \leqslant a_0^{K_2} 2^{4(1 + \log_2 K_1)} \max(p_{r_n}, q_{r_n})^{K_2 + 2(1 + \log_2 K_1)}.$$

Thus using this expression and $p_{r_n} < 2a_0 q_{r_n}$ in (19) and putting

$$c_2 = 2^{4m(1 + \log_2 K_1) + m(2 + K_2 + 2\log_2 K_1)} 3^{3m} a_0^{2mK_2 + 2m(1 + \log_2 K_1)} c_1^m,$$

$$c_3 = 1 + m(2 + K_2 + 2\log_2 K_1) + m^2$$

we get

$$H(\alpha_{r_n}) \leqslant c_2 q_{r_n}^{c_3 - 1} \qquad (n = 0, 1, \ldots).$$

Since $q_{r_n} \to \infty$ as $n \to \infty$, there exists a positive integer n_1 such that if $n > n_1$ then

$$H(\alpha_{r_n}) \leqslant q_{r_n}^{c_3} \qquad (c_3 > 0). \tag{20}$$

Now, to prove that $\xi \in \bigcup_{j=1}^m U_j$, we shall approximate ξ by the algebraic numbers α_{r_n} $(n = 0, 1, \ldots)$.

By the definitions of ξ and α_{r_n}, we see that

$$|\xi - \alpha_{r_n}| \leqslant \frac{1}{(q'_{s_n})^2}. \tag{21}$$

Now we put

$$\frac{R_{(r_n, s_n)}}{S_{(r_n, s_n)}} = \left\langle a_{r_n + 1}, a_{r_n + 2}, \ldots, a_{s_n} \right\rangle.$$

By Lemma III we can see easily that

$$q'_{s_n} > S_{(r_n, s_n)} \tag{22}$$

and

(23) $$q_{s_n} = q_{r_n} R_{(r_n, s_n)} + q_{r_n - 1} S_{(r_n, s_n)} \leq 2 q_{r_n} \max\left(R_{(r_n, s_n)}, S_{(r_n, s_n)}\right)$$

or using the relations $R_{(r_n, s_n)} \leq 2 a_{r_n + 1} S_{(r_n, s_n)}$ and (22) in (23) we get

(24) $$q_{s_n} \leq 4 a_{r_n + 1} q_{r_n} q'_{s_n}.$$

Now we shall give an upper bound for $a_{r_n + 1}$. By applying Lemma II with $\alpha_1 = \alpha$, $\alpha_2 = p_{r_n}/q_{r_n}$ $(n > n_1)$ and putting

$$c_4 = 2^{3m-1} a_0^m (m+1) H(\alpha)$$

we obtain

(25) $$\left|\alpha - \frac{p_{r_n}}{q_{r_n}}\right| \geq \frac{1}{c_4 q_{r_n}^m} \qquad (n > n_1)$$

(that is, we obtain Liouville's Theorem).

On the other hand it follows from the theory of continued fractions that

(26) $$\left|\alpha - \frac{p_{r_n}}{q_{r_n}}\right| \leq \frac{1}{a_{r_n + 1} q_{r_n}^2}.$$

Finally combining the relations (24), (25), (26) we get

(27) $$q_{s_n} \leq 4 c_4 q_{r_n}^{m-1} q'_{s_n} \qquad (n > n_1).$$

Hence the relation (27) and condition (3a) show that there exists a positive integer n_2 such that

(28) $$q_{s_n} \leq \left(q'_{s_n}\right)^2$$

holds if $n \geq \max(n_1, n_2)$.

Finally using (20), (28) and condition (3a) in (21) we obtain

(29) $$|\xi - \alpha_{r_n}| \leq \frac{1}{\left(q'_{s_n}\right)^2} \leq \frac{1}{q_{s_n}} \leq \left(\left(H(\alpha_{r_n})\right)\frac{\log q_{s_n}}{c_3 \cdot \log q_{r_n}}\right)^{-1} \qquad (n \geq \max(n_1, n_2)).$$

Since $\lim_{n \to \infty}(\log q_{s_n}/\log q_{r_n}) = \infty$, (29) shows that $\xi \in \bigcup_{j=1}^m U_j$.

We shall complete the proof by showing that $\xi \notin \bigcup_{j=1}^{m-1} U_j$. Let β be an algebraic number of degree f $(0 < f \leq m-1)$. Since $m \neq f$ we can apply Lemma II with $\alpha_1 = \beta$, $\alpha_2 = \alpha_{r_n}$ $(n \geq \max(n_1, n_2))$ and we get

(30) $$|\beta - \alpha_{r_n}| \geq \frac{1}{c_5 H(\beta)^m H(\alpha_{r_n})^{m-1}} \qquad (n \geq \max(n_1, n_2)),$$

where $c_5 = 2^{m-1} m^m (m+1)^{m-1}$ is a positive constant. Next using (20) in (30) and putting $c_6 = c_3(m-1)$

(31) $$|\beta - \alpha_{r_n}| \geq \frac{1}{c_5 H(\beta)^m (q_{r_n})^{c_6}} \qquad (n \geq \max(n_1, n_2)).$$

On the other hand it follows from the condition (3b) that there exists a positive real number T_0 such that

$$q_{s_n}^{T_0} \geqslant q_{r_{n+1}} \qquad (n \geqslant \max(n_1, n_2)). \tag{32}$$

Thus, using the relations (21), (28), (31), (32) in the inequality

$$|\xi - \beta| \geqslant |\beta - \alpha_{r_n}| - |\xi - \alpha_{r_n}|$$

we obtain that

$$|\xi - \beta| \geqslant \frac{1}{c_5 H(\beta)^m (q_{r_n})^{c_6}} - \frac{1}{(q_{r_{n+1}})^{1/T_0}} \qquad (n \geqslant \max(n_1, n_2)). \tag{33}$$

Suppose that

$$H(\beta) \geqslant \max\left(q_{r_{\max(n_1,n_2)}}, 2c_5\right). \tag{34}$$

It is clear that, for every $H(\beta)$ with (34), there exists a positive integer j ($j \geqslant \max(n_1, n_2)$) such that

$$q_{r_j} \leqslant H(\beta) < q_{r_{j+1}}. \tag{35}$$

Now we consider two cases in (35) as follows.

$$\text{(36)} \quad \begin{aligned} &\text{(a)} \quad q_{r_j} \leqslant H(\beta) < q_{r_{j+1}}^{1/(T_0(c_6+m+1))}, \\ &\text{(b)} \quad q_{r_{j+1}}^{1/(T_0(c_6+m+1))} \leqslant H(\beta) < q_{r_{j+1}}. \end{aligned}$$

Case 1. If $H(\beta)$ satisfies condition (36a), taking $n = j$ in (33) and using (34), (36a) we obtain

$$|\xi - \beta| \geqslant \frac{1}{c_5 H(\beta)^{m+c_6}} - \frac{1}{H(\beta)^{m+c_6+1}} \geqslant \frac{1}{2c_5 H(\beta)^{m+c_6+1}}. \tag{37}$$

Case 2. Suppose that (36b) holds. Then taking $n = j + 1$ in (32) and using the first part of (36b) we obtain that

$$|\xi - \beta| \geqslant \frac{1}{c_5 H(\beta)^{m+c_6T_0(c_6+m+1)}} - \frac{1}{(q_{r_{j+2}})^{1/T_0}}$$

or

$$|\xi - \beta| \geqslant \frac{1}{c_5 H(\beta)^{m+c_6T_0(c_6+m+1)}} - \frac{1}{H(\beta)^{(\log q_{r_{j+2}}/\log q_{r_{j+1}})\cdot(1/T_0)}}. \tag{38}$$

It is easy to see that condition (3a) in the Theorem implies that

$$\lim_{j\to\infty} \frac{\log q_{r_{j+2}}}{\log q_{r_{j+1}}} = \infty.$$

So using this relation in (38), we get

$$|\xi - \beta| \geqslant \frac{1}{2c_5 H(\beta)^{m+c_6T_0(c_6+m+1)}} \tag{39}$$

for sufficiently large j.

Thus the relations (37), (39) give as $\xi \notin \bigcup_{j=1}^{m-1} U_j$ and this completes the proof of the Theorem.

Note that it can be seen easily from the proof that if we replaced the condition (3a) by

$$\overline{\lim_{n\to\infty}} \frac{\log q_{s_n}}{\log q_{r_n}} = \infty, \qquad \underline{\lim_{n\to\infty}} \frac{\log q_{s_n}}{\log q_{r_n}} \geqslant m + c_6 T_0(c_6 + m + 1) + 1,$$

the theorem is still true. (Of course, $m + c_6T_0(c_6 + \mathrm{m} + 1) + 1$ is effectively computable.)

As a special case of the Theorem we take $r_{n+1} = s_n + 2$ and we define integers b_j $(j = 0, 1, \ldots)$ by

$$b_j = \begin{cases} a_j, & j \neq r_{n+1} - 1 \quad (n = 0, 1, \ldots), \\ a_j + 1, & j = r_{n+1} - 1 \quad (n = 0, 1, \ldots). \end{cases}$$

By the Theorem, we have

$$\xi = \langle b_0, b_1, \ldots, b_m, \ldots \rangle \in U_m.$$

Hence it follows from the Thue-Siegel-Roth Theorem and the above example that

COROLLARY. *For every positive integer m $(m > 1)$ there exists a subset K_m of U_m which has the continuum cardinality such that if $\xi \in K_m$ and $\varepsilon > 0$, then*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

has only finitely many solutions in integer p, q.

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