MINIMIZING SETUPS FOR CYCLE-FREE ORDERED SETS

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Abstract. A machine performs a set of jobs one at a time subject to a set of precedence constraints. We consider the problem of scheduling the jobs to minimize the number of "setups".

Suppose a single machine is to perform a set of jobs, one at a time; a set of precedence constraints prohibits the start of certain jobs until some other jobs are already completed. Any job which is performed immediately after a job which is not constrained to precede it, however, requires a "setup"—entailing some fixed additional cost. The problem is schedule the jobs to minimize the number of setups.

It is common to render "a set of precedence constraints on a set of jobs" as "an antisymmetric and transitive binary relation on a set," that is, "a (partial) ordering on a set." In this analogy a "schedule satisfying the precedence constraints" becomes "a linear extension of the ordered set" (of all jobs). The problem of minimizing the number of setups can be entirely recast as a problem concerning linear extensions of an ordered set. The problem itself is attributed in [2] to Kuntzmann (cf. [6]). Progress on the problem can be found in several papers including [3, 4, and 7] and recently W. R. Pulleyblank [7] has shown that this problem belongs to that class of problems whose complexity is described as NP-hard.

For elements a, b of an ordered set (P, \( \leq \))—simply written as P—we say that b covers a if a \( \leq b \) in P and a \( \leq c < b \) implies a = c. Let L be a linear extension of P; that is, a total ordering of the underlying set of P such that a < b in L whenever a < b in P. A 'setup for L' is an ordered pair (a, b) of elements of P for which b covers a in L but a \( \not< b \) (and hence also a \( \not< b \)) in P. Let \( s_L(P) \) count the number of such ordered pairs and let

\[
s(P) = \min \left\{ s_L(P) \mid L \text{ is a linear extension of P} \right\}.
\]

The problem is construct a linear extension L of the ordered set P for which \( s_L(P) = s(P) \).

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Any linear extension $L$ of $P$ can be obtained by partitioning $P$ into chains (linearly ordered subsets) $C_1, C_2, \ldots, C_m$ such that $x < y$ in $L$ if either $x < y$ in $P$, or $x \in C_i$ and $y \in C_j$, where $i < j$. In particular, $L$ is the \textit{linear sum} of chains $L = C_1 \oplus C_2 \oplus \cdots \oplus C_m$.

If the greatest element $\max(C_i)$ of $C_i$ is not below the least element $\min(C_{i+1})$ of $C_{i+1}$ in $P$, then $(\max(C_i), \min(C_{i+1}))$ is a setup for $L$. Evidently, $s_L(P) \leq m - 1$ and if $\max(C_i) \not< \min(C_{i+1})$ for each $i = 1, 2, \ldots, m - 1$, then $s_L(P) = m - 1$. According to Dilworth's theorem [5], the smallest number of chains into which $P$ can be partitioned is equal to the \textit{width} $w(P)$ of $P$—the size of a maximum-sized antichain. Therefore, $s(P) \geq w(P) - 1$.

Of course, equality does not in general obtain. Indeed, a partition $C_1, C_2, \ldots, C_{w(P)}$ of $P$ into chains can be arranged to form a linear extension of $P$ only if there is a permutation $\rho$ of $\{1, 2, \ldots, w(P)\}$ such that $\rho(i) < \rho(j)$ implies $x \nless y$ for any $x \in C_{\rho(i)}$ and $y \in C_{\rho(j)}$. No such permutation could exist if there were a subset (say, $\{C_1, C_2, \ldots, C_n\}$) of the partition, and elements $x_i, y_i \in C_i$, $i = 1, 2, \ldots, n$, satisfying

$$y_1 < x_1, x_1 > y_2, y_2 < x_2, x_2 > y_3, \ldots, x_{n-1} > y_n, y_n < x_n, x_n > y_1.$$  

An ordered set $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ of size $2n$, $n \geq 2$, with these comparabilities, and no others, is called an \textit{alternating $2n$-cycle}, or more briefly a $2n$-cycle (see Figure 2).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Figure 2}
\end{figure}

The ordered sets shown in Figure 3 are cycle-free, that is, contain no subset isomorphic to an alternating $2n$-cycle.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Figure 3}
\end{figure}
The principal result of this paper is

**Theorem.** Let \( P \) be an ordered set without alternating cycles. Then \( s(P) = w(P) - 1 \).

The case where \( P \) has length two (that is, \( P \) has no three-element chain) is particularly easy to verify. We proceed by induction on the size of \( P \): if \( P \) contains an isolated element \( a \) then \( w(P - \{ a \}) = w(P) - 1 \) and clearly \( s(P) = s(P - \{ a \}) + 1 \). Otherwise, as \( P \) is cycle-free there is an element \( b \) comparable with precisely one other element, say, \( b < c \). Again if \( w(P - \{ b \}) = w(P) - 1 \) then the induction hypothesis applies; otherwise, \( w(P - \{ b \}) = w(P) \) and, indeed, \( w(P - \{ b, c \}) = w(P) - 1 \). Finally, \( s(P) = s(P - \{ b, c \}) + 1 \), so in any case, \( s(P) = w(P) - 1 \).

Before we turn to the proof of the theorem, note from the ordered sets illustrated in Figure 4 that the converse of the theorem cannot hold.

\[
\begin{align*}
\text{Figure 4} \\
s(P_1) &= 2 = w(P_1) - 1 \quad (i = 1, 2)
\end{align*}
\]

**Proof of the theorem.** We proceed by induction on \( m = w(P) \). Let \( C_1, C_2, \ldots, C_m \) be a sequence of maximal chains of \( P \) such that

\[
P = \bigcup_{i=1}^{m} C_i.
\]

(Such a sequence can always be obtained by extending each of the \( m \) chains in a partition of \( P \) by width-many chains.)

Let \( x, y, z \in C_i \) with \( x < y < z \) and suppose that for some \( j, \{x, y, z\} \cap C_j = \{y\} \). Then some element \( x' \) in \( C_j \) must be noncomparable to \( x \), else the addition of \( x \) would extend \( C_j \); similarly there must be an element \( z' \) of \( C_j \) noncomparable to \( z \). But then \( \{x, z, x', z'\} \) is a 4-cycle, contradicting the hypothesis of the theorem. It follows that, for any \( i \) and \( j \) and any \( y \in C_i \cap C_j \), either \( \{x \in C_i \cup C_j \mid x < y\} \) is a chain or \( \{z \in C_i \cup C_j \mid z > y\} \) is a chain.

For each \( i \), let

\[
P_i = C_i - \bigcup_{j \neq i} C_j.
\]

Then \( P_i \neq \emptyset \) for each \( i = 1, 2, \ldots, m \), for otherwise \( m = w(P) < m \). We now introduce a binary relation \( \rightarrow \) on \( \{C_i \mid i = 1, 2, \ldots, m\} \) as follows: \( C_i \rightarrow C_j \) if there are elements \( x \in P_i \) and \( y \in C_j - C_i \) such that \( x > y \) in \( P \). The definition is motivated by this observation:

if for some \( i \), \( C_i \rightarrow C_j \) for all \( j \) then \( s(P) = w(P) - 1 \).
To prove this let $x = \max(P_i)$, $C = \{y \in C_i \mid y \leq x\}$, and let $P' = P - C$. Then $w(P') = w(P) - 1$ and by the induction hypothesis there is a linear extension $L'$ of $P'$ consisting of a linear sum of $m - 1$ chains of $P'$. We claim $L = C \oplus L'$ is a linear extension of $P$; if not, there are elements $y \in C$ and $z \in P' \cap C_j$, for some $j \neq i$, with $y > z$. Hence $z < x$ and since $C_i \leftrightarrow C_j$, it must be that $z \in C_i$; then $z \in C$, an impossibility.

We may therefore suppose that for each $i$ there is some $j$ such that $C_i \rightarrow C_j$. After suitable relabelling, there is a sequence $1, 2, \ldots, n$ of smallest length such that $C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_n \rightarrow C_1$.

Choose $x_i \in P_i$ and $y_i \in C_i - C_{i-1}$ with $x_i > y_{i+1}$, for each $i = 1, 2, \ldots, n$ (mod $n$). Observe that $x_i > y_i$ for each $i$, $1 \leq i \leq n$. We conclude the proof by verifying that \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\} must now contain an alternating cycle. Let us suppose that it is not itself a 2n-cycle.

Case (i). Let $x_i > x_j$. Since $x_i \in C_j$ there is some $x > x_j$ in $C_j$ which is incomparable with $x_i$. Further, since $y_{j+1} \notin C_j$ there is some $y < x_j$ in $C_j$ which is incomparable with $y_{j+1}$; then $(x_i, x, y_{j+1}, y)$ is a 4-cycle.

Case (ii). Let $y_i = y_j, i \neq j$. Then $C_{i-1} \rightarrow C_j$, contradicting the minimality of $n$.

Case (iii). Let $y_i < y_j$. If $y_i \in C_j$ then there is $y < y_j$ in $C_j$ noncomparable with $y_i$, so $(x_{j-1}, x_j, y_j, y)$ is a 4-cycle. If $y_i \in C_j$ then $C_{i-1} \rightarrow C_j$, again contradicting the minimality of $n$.

It follows that $y_i$ is noncomparable with $y_j$ for each $i \neq j$.

Case (iv). Let $x_i > y_j$, where $j \neq i$ and $j \neq i + 1$. Since $y_j$ is noncomparable with $y_i$, $y_j \notin C_i$ so $C_i \rightarrow C_j$ which is again impossible.

Case (v). Let $x_i < y_j$. Then $y_i < y_j$ which was already ruled out.

This completes the proof.

An algorithm. Implicit in the proof of the theorem is an algorithm to construct a linear extension $L$ of a cycle-free ordered set $P$ which is optimal in the sense that $s_L(P) = s(P) = w(P) - 1$. The following procedure, though inductive, is based on a single covering $C_1, C_2, \ldots, C_{w(P)}$ of $P$ by maximal chains.

**Figure 5**
According to the proof of the theorem, in any such covering there is a chain (say, $C_i$) such that for any $i = 2, 3, \ldots, w(P)$, $C_i \Rightarrow C_i$. Let $c_i = \max(P_i)$, $C'_i = \{x \in C_i \mid x \leq c_i\}$, and $Q = P - C'_i$. Then $Q$ is covered by the chains $Q \cap C_2, \ldots, Q \cap C_{w(P)}$, and by inductive use of this algorithm $Q$ has a linear extension 

$$L' = C'_2 \oplus C'_3 \oplus \cdots \oplus C'_{w(P)}$$

with $s_L(Q) = w(Q) - 1$, where $C'_i \subset C_i$ for each $i = 2, 3, \ldots, w(P)$. Then $L = C'_1 \oplus L'$ is a linear extension of $P$ for which $s_L(P) = w(Q) = w(P) - 1$ as required.

The algorithm is illustrated in Figure 5 for a particular cycle-free ordered set of width three.

REFERENCES


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