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A CRITERION FOR FINITE MODULE TYPE

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ABSTRACT. The following result is proved: If a p-block of a finite group has only finitely many indecomposable liftable modules with the defect group of the block as a vertex or if it has only finitely many indecomposable periodic modules, then the block is of finite module type.

1. Introduction. The purpose of this paper is to give a criterion for a block to be of finite module type (that is, the block has only finitely many nonisomorphic indecomposable FG-modules) by looking at some nice families of modules in the block, namely the liftable and the periodic modules.

Let G be a finite group and (F, R, K) a p-modular splitting system for G, that is, R is a complete discrete valuation ring with quotient field K of characteristic 0 and residue class field F of characteristic p, such that K and F are splitting fields for all subgroups of G. For further notation we refer to **[3]**. Then we have the following criterion:

THEOREM. *Let B be a block of FG with defect group D. Then the following are equivalent*:

(i) *B is of finite module type.*

(ii) *B has only finitely many indecomposable liftable modules with vertex D.*

(iii) *B has only finitely many indecomposable periodic modules.*

Of course, "finitely many" always means up to isomorphism.

2. Preliminary results on periodicity. In the following $A \in \{R, F\}$ and for $A = R$ an AG-module means an RG-lattice, that is an RG-module which is finitely generated and torsionfree as an R-module. Moreover, in this chapter F and K need not be splitting fields.

First let us recall the definition of Heller's operator Ω. If U is an AG-module then by **[4, 6]** there exists a minimal projective presentation $0 \to \Omega U \to P \to U \to 0$, that is, P is a projective AG-module and ΩU has no projective summands. P and ΩU are uniquely determined by U (up to isomorphism), and if U is indecomposable and nonprojective, then so is ΩU. By iterating this process we get $\Omega^i U$, for all $i \in \mathbf{N}$. Then an AG-module U is called periodic if $U \simeq \Omega^i U \oplus L$ for some $i > 0$ and projective AG-module L.

A block B of AG is called periodic if all modules in B are periodic.

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2.1 REMARK. Since a direct sum of modules is periodic iff all summands are periodic, a block B is periodic iff the indecomposable modules in B are periodic.

The following fact is well known and can easily be proved.

2.2 LEMMA. *An AG-module is periodic iff its source is periodic.*

2.3 PROPOSITION. *A block B of AG is periodic iff its defect group D is cyclic or generalized quaternion.*

PROOF. If $A = F$ then by [2] there is an indecomposable FG-module U in B with source F_D. If $A = R$, then using this for the corresponding block of FG and [8] we thus also have an indecomposable RG-lattice U in B with source R_D. Now if B is periodic then so is U and hence A_D by 2.2. But this is well known to be equivalent to D being cyclic or generalized quaternion. If D is cyclic or quaternion then all AD-modules are periodic. Hence all modules in B are direct summands of induced periodic modules, and so B is periodic.

3. Proof of the theorem.

3.1 PROPOSITION. *Let F be any field of characteristic p and B be a block of FG with defect group D, which has only finitely many indecomposable liftable FG-modules with vertex D. Then B is periodic.*

PROOF. As in the preceding proof there is an indecomposable module M in B with source F_D which is liftable, say $M \simeq \overline{U}$, where U is an indecomposable RG-lattice. Now $\Omega^i(M) \simeq \overline{\Omega^i(U)}$ for all $i \in \mathbb{N}_0$ by [4], so the $\Omega^i(M)$ are indecomposable liftable modules with vertex D in B. By assumption there are only finitely many such modules in B, hence M must be periodic and by 2.2 so is F_D. Now this implies B is periodic by 2.3.

3.2 PROPOSITION. *Let B be a block with generalized quaternion defect group Q. Then B has infinitely many indecomposable liftable modules with vertex Q.*

PROOF. Let t be the unique involution in Q. First suppose $\langle t\rangle \lhd G$. Then $G = C_G(\langle t\rangle)$. By [3, 64.5] we have a uniquely determined block $\overline{B}$ of $\overline{G} = G/\langle t\rangle$ corresponding to B (by $\overline{B} \subseteq B$) with defect group $Q/\langle t\rangle$. Since $Q/\langle t\rangle$ is neither cyclic nor quaternion, 2.3 together with 3.1 shows that $\overline{B}$ has infinitely many indecomposable liftable modules with vertex $Q/\langle t\rangle$. By regarding these $F\overline{G}$-modules as FG-modules we obtain an infinite number of indecomposable liftable modules with vertex Q in B.

In the general case we use Green correspondence between $N_G(\langle t\rangle)$ and G. As $\langle t\rangle$ is a trivial intersection set, the Green correspondent of a liftable module is again liftable (this is easy, see e.g. [7, 2.8]). So by using the first case and Green correspondence we get an infinite set of nonisomorphic indecomposable liftable modules with vertex Q in B.

3.3 PROOF OF THE THEOREM.

(i) $\Rightarrow$ (ii), (iii) is trivial.

(ii) $\Rightarrow$ (i) As a direct consequence of the preceding propositions D is cyclic; hence B is of finite module type by [5].

(iii) ⇒ (i) By [5] a block is of finite module type iff its defect group is cyclic. So suppose D is not cyclic. Then by [1] there are infinitely many indecomposable periodic FD-modules. Hence there are infinitely many indecomposable periodic FY-modules which are sources, for some $Y \leqslant D$, since a source is a source for only finitely many FD-modules. By [2] each of these sources is a source for an indecomposable module in B, which is periodic by 2.2. Since every module has only finitely many sources we thus have infinitely many indecomposable periodic modules in B.

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