ISOMETRIES OF $A_C(K)$

T. S. S. R. K. RAO

Abstract. We completely describe isometries of $A_C(K)$, when $K$ is a compact Choquet simplex, using facially continuous functions on the extreme boundary.

1. Introduction. Let $K$ be a compact convex set in a locally convex space and denote by $E(K)$ the set of extreme points of $K$ and by $A_C(K)$ the continuous complex-valued affine functions on $K$, equipped with the supremum norm.

We first describe a class of isometries for $A_C(K)$ when $K$ is any compact convex set and give a sufficient condition on an isometry, in terms of facially continuous functions on $E(K)$, so that the isometry in question is in the prescribed class and then deduce that when $K$ is a Choquet simplex, the class of isometries considered, completely describes the isometries of $A_C(K)$.

2. Notations and definitions. For the concepts and results of convexity theory used here we cite [1].

A set $D \subseteq E(K)$ is said to be facially closed if there exists a closed split face $F$ of $K$ such that $E(F) = D$. The sets $D$ form the closed sets of a topology on $E(K)$ called the facial topology.

Let $C$ denote the complex plane and $\Gamma$, the unit circle in $C$. For a probability measure $\mu$, let $r(\mu)$ denote the resultant of $\mu$ and $\text{Supp } \mu$ denote the topological support of $\mu$.

3. Description of isometries. Following the notations of [1], we denote by $Z(A_C(K))$ the set of elements $b \in A_C(K)$ such that for every $a \in A_C(K)$ there exists $c \in A_C(K)$ satisfying

$$c(x) = a(x) \cdot b(x) \quad \forall x \in E(K).$$

Since for any $b \in Z(A_C(K))$, real and imaginary parts of $b$ are in $Z(A(K))$, using Corollary II.7.4 and Theorem II.7.10 of [1], we can easily see that for $b \in A_C(K)$, $b$ is in $Z(A_C(K))$ if and only if $b | E(K) \to C$ is continuous in the facial topology.

Let $Q: K \to K$ be an onto affine homeomorphism and let $a_0 \in Z(A_C(K))$ be such that $|a_0| = 1$ on $E(K)$. Define $\Phi: A_C(K) \to A_C(K)$ by $\Phi(a) = c$, where $c$ is the unique element of $A_C(K)$ such that $c(x) = a(Q(x)) \cdot a_0(x) \forall x \in E(K)$.

It is easy to see that $\Phi$ is an onto isometry and $\Phi(1) = a_0$. 

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Theorem 3.1. Let \( \Phi: A_c(K) \to A_c(K) \) be any onto isometry. Assume
\[
\Phi(1) \in Z(A_c(K)).
\]
Then there exists an affine homeomorphism \( Q \) from \( K \) onto \( K \) such that
\[
\Phi(a)(x) = a(Q(x))\Phi(1)(x) \quad \forall x \in E(K).
\]

Proof. Define \( \delta: K \to A(K)^* \) by \( \delta(x)(a) = a(x) \) \( \forall a \in A_c(K) \) and \( x \in K \). It is well known that \( \delta \) is an affine homeomorphism of \( K \) onto \( \{ f \in A_c(K)^*: \| f \| = \| f(1) \| = 1 \} \), with \( w^* \)-topology. Since \( \Phi^*: A_c(K)^* \to A_c(K)^* \) is an onto isometry and a \( w^* \)-homeomorphism it is easy to see that \( \Phi^*(\delta(E(K))) \subseteq \Gamma \cdot \delta(E(K)) \).

Let \( x \in E(K) \). Since \( A_c(K) \) separates points of \( K \) and \( 1 \in A_c(K) \), there exist unique \( x' \in E(K) \) and \( t \in \Gamma \), such that \( \Phi^*(\delta(x)) = t \cdot \delta(x') \). Moreover
\[
\Phi^*(\delta(x))(1) = \delta(x)(\Phi(1)) = \Phi(1)(x) = t.
\]
Hence \( \Phi(1) \) is of modulus 1 on \( E(K) \). Let \( \Phi(1) = u + iv \), \( u, v \in A(K) \) (real-valued functions in \( A_c(K) \)). Then since \( Z(A_c(K)) \) is selfadjoint, \( \Phi(1) = u - iv \) is in \( Z(A_c(K)) \). Define now \( T: A_c(K) \to A_c(K) \) by
\[
T(a)(x) = \Phi(a)(x) \cdot \Phi(1)(x) \quad \forall x \in E(K).
\]
Since \( |\Phi(1)| = 1 \) on \( E(K) \), it follows from the remarks in the beginning of this section that \( T \) is a well-defined, onto isometry. Moreover, \( T(1) = 1 \). It is easy to see that \( T^* \) maps \( \delta(K) \) onto \( \delta(K) \) and \( Q = \delta^{-1} \circ T^* \circ \delta \) is an affine homeomorphism of \( K \) onto \( K \). That \( \Phi(a)(x) = a(Q(x)) \cdot \Phi(1)(x) \) \( \forall x \in E(K) \) follows from (*) and the definition of \( T \).

Definition (Effros). Say a closed set \( D \subseteq K \) is a dilated set if for any maximal measure \( \mu \) with \( r(\mu) \subseteq D \), \( \text{Supp} \ \mu \subseteq D \).

Proposition 3.2. Let \( K \) be a compact Choquet simplex and let \( a_0 \in A_c(K) \) and \( |a_0| = 1 \) on \( E(K) \). Then \( a_0 \in Z(A_c(K)) \).

Proof. In view of the results quoted in the beginning of this section it is sufficient to show that \( a_0 | E(K) \) is facially continuous.

For a closed set \( B \subseteq T \), let \( B' = \{ x \in \overline{E(K)}: a_0(x) \in B \} \). We claim that the closed set \( B' \) is a dilated set. Let \( \mu \) be a maximal probability measure with \( x_0 = r(\mu) \subseteq B' \). Since
\[
1 = |a_0(x_0)| \leq \int_{E(K)} a_0 \, d\mu = \int_{E(K)} |a_0| \, d\mu \leq 1,
\]
we get that \( a_0 \equiv a_0(x_0) \) on \( \text{Supp} \ \mu \) and hence \( \text{Supp} \ \mu \subseteq B' \).

It now follows from a result of [2] that \( F \), the closed convex hull of \( B' \), is a split face and hence \( \{ x \in E(K): a_0(x) \in B \} = F \cap E(K) \) is a facially closed set.

Remark. When \( K \) is a simplex, \( a \in A_c(K) \) is an extreme point of the closed unit ball of \( A_c(K) \) iff \( |a| = 1 \) on \( E(K) \) iff \( a \in Z(A_c(K)) \) and is an extreme point of the closed unit ball of \( Z(A_c(K)) \).

Corollary 3.3. If \( K \) is a compact Choquet simplex then for any onto isometry \( \Phi \) of \( A_c(K) \), \( \exists \) an affine homeomorphism \( Q \) of \( K \) such that
\[
\Phi(a)(x) = a(Q(x)) \cdot \Phi(1)(x) \quad \forall x \in E(K).
\]
Proof. We have observed in the proof of Theorem 3.1 that $|\Phi(1)| = 1$ on $E(K)$, hence the conclusion follows from Corollary 3.2 and Theorem 3.1.

Remark. These results generalize the classical Banach-Stone theorem dealing with the isometries of $C_c(X)$, where $X$ is a compact Hausdorff space; also generalized is the work of A. J. Lazar [3] on isometries of $A(K)$.

4. Example. We end by giving a simple example of a nonsimplicial compact convex set $K$ and an isometry $\Phi$ of $A_c(K)$ which is not of the form described in Theorem 3.1.

Let $K$ be the unit square in $\mathbb{R}^2$ centred at $(0,0)$, so

$$E(K) = \{(x, y) : |x| = 1 = |y|\} \cdot K$$

has no proper split faces and hence $Z(A_c(K)) = \{\alpha \cdot 1 : \alpha \in \mathbb{C}\}$. Any $f \in A_c(K)$ is of the form $f(x, y) = ax + by + c$ where $a, b, c \in \mathbb{C}$. Define $\Phi(f)(x, y) = cx + by + a$. Now $\|f\| = \max |a \pm b \pm c|$ and $\|\Phi(f)\| = \max |c \pm b \pm a|$ hence $\Phi$ is an isometry. It is obvious that $\Phi$ is onto. But $\Phi(1) = x$, a nonconstant. Hence $\Phi$ is not of the form in Theorem 3.1.

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References