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ISOMETRIES OF $A_{\mathbf{C}}(K)$

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ABSTRACT. We completely describe isometries of $A_{\mathbf{C}}(K)$, when K is a compact Choquet simplex, using facially continuous functions on the extreme boundary.

1. Introduction. Let K be a compact convex set in a locally convex space and denote by $E(K)$ the set of extreme points of K and by $A_{\mathbf{C}}(K)$ the continuous complex-valued affine functions on K, equipped with the supremum norm.

We first describe a class of isometries for $A_{\mathbf{C}}(K)$ when K is *any* compact convex set and give a sufficient condition on an isometry, in terms of facially continuous functions on $E(K)$, so that the isometry in question is in the prescribed class and then deduce that when K is a Choquet simplex, the class of isometries considered, completely describes the isometries of $A_{\mathbf{C}}(K)$.

2. Notations and definitions. For the concepts and results of convexity theory used here we cite [**1**].

A set $D \subset E(K)$ is said to be facially closed if there exists a closed split face F of K such that $E(F) = D$. The sets D form the closed sets of a topology on $E(K)$ called the facial topology.

Let $\mathbf{C}$ denote the complex plane and Γ, the unit circle in $\mathbf{C}$. For a probability measure μ, let $r(\mu)$ denote the resultant of μ and $\operatorname{Supp}\mu$ denote the topological support of μ.

3. Description of isometries. Following the notations of [**1**], we denote by $Z(A_{\mathbf{C}}(K))$ the set of elements $b \in A_{\mathbf{C}}(K)$ such that for every $a \in A_{\mathbf{C}}(K)$ there exists $c \in A_{\mathbf{C}}(K)$ satisfying

$$c(x) = a(x) \cdot b(x) \qquad \forall x \in E(K).$$

Since for any $b \in Z(A_{\mathbf{C}}(K))$, real and imaginary parts of b are in $Z(A(K))$, using Corollary II.7.4 and Theorem II.7.10 of [**1**], we can easily see that for $b \in A_{\mathbf{C}}(K)$, b is in $Z(A_{\mathbf{C}}(K))$ if and only if $b \mid E(K) \to \mathbf{C}$ is continuous in the facial topology.

Let Q: $K \to K$ be an onto affine homeomorphism and let $a_0 \in Z(A_{\mathbf{C}}(K))$ be such that $|a_0| = 1$ on $E(K)$. Define Φ: $A_{\mathbf{C}}(K) \to A_{\mathbf{C}}(K)$ by $\Phi(a) = c$, where c is the unique element of $A_{\mathbf{C}}(K)$ such that $c(x) = a(Q(x)) \cdot a_0(x)\ \forall x \in E(K)$.

It is easy to see that Φ is an onto isometry and $\Phi(1) = a_0$

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THEOREM 3.1. *Let Φ: $A_{\mathbf{C}}(K) \to A_{\mathbf{C}}(K)$ be any onto isometry. Assume*

$$\Phi(1) \in Z(A_{\mathbf{C}}(K)).$$

Then there exists an affine homeomorphism Q from K onto K such that

$$\Phi(a)(x) = a(Q(x))\Phi(1)(x) \qquad \forall x \in E(K).$$

PROOF. Define δ: $K \to A(K)^*$ by $\delta(x)(a) = a(x)$ $\forall a \in A_{\mathbf{C}}(K)$ and $x \in K$. It is well known that δ is an affine homeomorphism of K onto $\{f \in A_{\mathbf{C}}(K)^*: \|f\| = f(1) = 1\}$, with w^*-topology. Since Φ^*: $A_{\mathbf{C}}(K)^* \to A_{\mathbf{C}}(K)^*$ is an onto isometry and a w^*-homeomorphism it is easy to see that $\Phi^*(\delta(E(K))) \subset \Gamma \cdot \delta(E(K))$.

Let $x \in E(K)$. Since $A_{\mathbf{C}}(K)$ separates points of K and $1 \in A_{\mathbf{C}}(K)$, there exist unique $x' \in E(K)$ and $t \in \Gamma$, such that $\Phi^*(\delta(x)) = t \cdot \delta(x')$. Moreover

$$(*) \qquad \Phi^*(\delta(x))(1) = \delta(x)(\Phi(1)) = \Phi(1)(x) = t.$$

Hence $\Phi(1)$ is of modulus 1 on $E(K)$. Let $\Phi(1) = u + iv$, $u, v \in A(K)$ (real-valued functions in $A_{\mathbf{C}}(K)$). Then since $Z(A_{\mathbf{C}}(K))$ is selfadjoint, $\overline{\Phi(1)} = u - iv$ is in $Z(A_{\mathbf{C}}(K))$. Define now T: $A_{\mathbf{C}}(K) \to A_{\mathbf{C}}(K)$ by

$$T(a)(x) = \Phi(a)(x) \cdot \overline{\Phi(1)}(x) \qquad \forall x \in E(K).$$

Since $|\Phi(1)| = 1$ on $E(K)$, it follows from the remarks in the beginning of this section that T is a well-defined, onto isometry. Moreover, $T(1) = 1$. It is easy to see that T^* maps $\delta(K)$ onto $\delta(K)$ and $Q = \delta^{-1} \circ T^* \circ \delta$ is an affine homeomorphism of K onto K. That $\Phi(a)(x) = a(Q(x)) \cdot \Phi(1)(x)$ $\forall x \in E(K)$ follows from $(*)$ and the definition of T.

DEFINITION (EFFROS). Say a closed set $D \subset K$ is a dilated set if for any maximal measure μ with $r(\mu) \in D$, $\operatorname{Supp} \mu \subseteq D$.

PROPOSITION 3.2. *Let K be a compact Choquet simplex and let $a_0 \in A_{\mathbf{C}}(K)$ and $|a_0| = 1$ on $E(K)$. Then $a_0 \in Z(A_{\mathbf{C}}(K))$.*

PROOF. In view of the results quoted in the beginning of this section it is sufficient to show that $a_0 | E(K)$ is facially continuous.

For a closed set $B \subset T$, let $B' = \{x \in \overline{E(K)}: a_0(x) \in B\}$. We claim that the closed set B' is a dilated set. Let μ be a maximal probability measure with $x_0 = r(\mu) \in B'$. Since

$$1 = |a_0(x_0)| = \left| \int_{E(K)} a_0 \, d\mu \right| \leq \int_{E(K)} |a_0| \, d\mu \leq 1,$$

we get that $a_0 \equiv a_0(x_0)$ on $\operatorname{Supp} \mu$ and hence $\operatorname{Supp} \mu \subset B'$.

It now follows from a result of [2] that F, the closed convex hull of B', is a split face and hence $\{x \in E(K): a_0(x) \in B\} = F \cap E(K)$ is a facially closed set.

REMARK. When K is a simplex, $a \in A_{\mathbf{C}}(K)$ is an extreme point of the closed unit ball of $A_{\mathbf{C}}(K)$ iff $|a| = 1$ on $E(K)$ iff $a \in Z(A_{\mathbf{C}}(K))$ and is an extreme point of the closed unit ball of $Z(A_{\mathbf{C}}(K))$.

COROLLARY 3.3. *If K is a compact Choquet simplex then for any onto isometry Φ of $A_{\mathbf{C}}(K)$, $\exists$ an affine homeomorphism Q of K such that*

$$\Phi(a)(x) = a(Q(x)) \cdot \Phi(1)(x) \qquad \forall x \in E(K).$$

PROOF. We have observed in the proof of Theorem 3.1 that $|\Phi(1)| = 1$ on $E(K)$, hence the conclusion follows from Corollary 3.2 and Theorem 3.1.

REMARK. These results generalize the classical Banach-Stone theorem dealing with the isometries of $C_{\mathbf{C}}(X)$, where X is a compact Hausdorff space; also generalized is the work of A. J. Lazar [3] on isometries of $A(K)$.

4. Example. We end by giving a simple example of a nonsimplicial compact convex set K and an isometry Φ of $A_{\mathbf{C}}(K)$ which is not of the form described in Theorem 3.1.

Let K be the unit square in $\mathbf{R}^2$ centred at (0, 0), so

$$E(K) = \{(x, y): |x| = 1 = |y|\} \cdot K$$

has no proper split faces and hence $Z(A_{\mathbf{C}}(K)) = \{\alpha \cdot 1: \alpha \in \mathbf{C}\}$. Any $f \in A_{\mathbf{C}}(K)$ is of the form $f(x, y) = ax + by + c$ where $a, b, c \in \mathbf{C}$. Define $\Phi(f)(x, y) = cx + by + a$. Now $\|f\| = \max|a \pm b \pm c|$ and $\|\Phi(f)\| = \max|c \pm b \pm a|$ hence Φ is an isometry. It is obvious that Φ is onto. But $\Phi(1) = x$, a nonconstant. Hence Φ is not of the form in Theorem 3.1.

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