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ON THE OSCILLATION AND NONOSCILLATION OF SECOND ORDER SUBLINEAR EQUATIONS

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ABSTRACT. An oscillation criterion and a nonoscillation criterion are given for the sublinear equation $y'' + a(t)|y|^\gamma \operatorname{sgn} y = 0$, $0 < \gamma < 1$, $t \in [0, \infty)$, where $a(t)$ is allowed to change sign. When applied to the special case $a(t) = t^\lambda \sin t$, we deduce oscillation for $\lambda > -\gamma$ and nonoscillation for $\lambda < -\gamma$.

We are interested in determining when all continuable solutions of the sublinear Emden-Fowler equation

$$y''(t) + a(t)|y(t)|^\gamma \operatorname{sgn} y(t) = 0, \qquad t \in [0, \infty), 0 < \gamma < 1, \tag{1}$$

are oscillatory. We are especially motivated by the particular case $a(t) = t^\lambda \sin t$ or more generally $t^\lambda f(t)$ where f is a periodic function of period T such that $\int_0^T f(t)\,dt \geqslant 0$.

We shall use as weight functions those $\phi\colon [0, \infty) \to [0, \infty)$ such that

$$\phi' > 0, \qquad \phi'' < 0. \tag{2}$$

In an earlier paper [2], the authors proved the following extension of the well-known Belohorec Theorem.

THEOREM. *If there exists a function ϕ satisfying* (2) *such that*

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T \int_0^t \phi^\gamma(\tau) a(\tau)\, d\tau\, dt = \infty,$$

then (1) *is oscillatory, i.e. all continuable solutions of* (1) *are oscillatory.*

An immediate consequence of this theorem is that when $a(t) = t^\lambda f(t)$ with $\int_0^T f(t)\,dt > 0$ and $\lambda \geqslant -\gamma$ or with $\int_0^T f(t)\,dt = 0$ and $\lambda > 1 - \gamma$, then (1) is oscillatory. When $\int_0^T f(t)\,dt = 0$ and $\lambda \leqslant 1 - \gamma$, the theorem fails to apply.

The first result of this paper is a sufficient oscillation condition that applies to cases in which

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^t \phi^\gamma(\tau) a(\tau)\, d\tau\, dt$$

exists and is finite. This condition allows us to deduce oscillation for $\lambda > -\gamma$ in our motivating example. To supplement our first result, we prove a nonoscillation

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theorem which allows us to settle the cases $\lambda < -\gamma$. The critical case $\lambda = -\gamma$ remains, however, unanswered. Thus the conjecture made by Butler in [**1**, p. 144] is almost completely proved. For reference to other known results consult [**2**] or [**3**].

We define the functions

$$\mathcal{A}(s) = \lim_{T\to\infty} \frac{1}{T}\int_s^T\int_s^t \phi^\gamma(\tau)a(\tau)\,d\tau\,dt \tag{3}$$

and

$$\mathcal{A}_+(s) = \max\{\mathcal{A}(s), 0\}.$$

Theorem 1. *If there is a weight function* ϕ *satisfying* (2) *so that the function* $\mathcal{A}$ *given by* (3) *is defined and satisfies*

$$\limsup_{T\to\infty}\left(\int_0^T \frac{A_+^2(s)}{s}\,ds\right)\left(\int_0^T \frac{\phi'^2(s)s}{\phi^2(s)}\,ds\right)^{-1} = \infty, \tag{4}$$

then (1) *is oscillatory.*

Proof. As in [**2**], the following easily verified identity plays a crucial role:

$$(\phi z^{\beta-1})'' + (\beta-1)\phi z^{\beta-3}z'^2 + \left(\frac{-\phi''}{\beta}\right)z^{\beta-1} = -\left(\frac{\beta-1}{\beta}\right)\phi^\gamma a, \tag{5}$$

where $z = (y/\phi)^\gamma$ and $\beta = 1/\gamma > 1$. Integrating (5) twice, first over $[s, t]$, then over $[s, T]$, we obtain

$$\begin{aligned} &\phi(T)z^{\beta-1}(T) - \phi(s)z^{\beta-1}(s) - (\phi(s)z^{\beta-1}(s))'(T-s) \\ &\quad + \int_s^T\int_s^t \left(\frac{-\phi''}{\beta}\right)z^{\beta-1}(\tau)\,d\tau\,dt \\ &\quad + (\beta-1)\int_s^T\int_s^t \phi(\tau)z^{\beta-3}(\tau)z'^2(\tau)\,d\tau\,dt \\ &= -\left(\frac{\beta-1}{\beta}\right)\int_s^T\int_s^t \phi^\gamma(\tau)a(\tau)\,d\tau\,dt. \end{aligned} \tag{6}$$

Dividing by T and letting $T\to\infty$, we see that, because the right-hand side tends to a limit and the integrands of the two integrals on the left-hand side as well as the first term are nonnegative, the following limits exist and are finite:

$$0 \leq \lim_{T\to\infty} \frac{\phi(T)z^{\beta-1}(T)}{T} = K < \infty, \tag{7}$$

$$0 \leq \lim_{T\to\infty} \frac{1}{T}\int_s^T\int_s^t \phi(\tau)z^{\beta-3}(\tau)z'^2(\tau)\,d\tau\,dt = G(s) < \infty \tag{8}$$

and

$$0 \leq \lim_{T\to\infty} \frac{1}{T}\int_s^T\int_s^t \left(-\frac{\phi''}{\beta}\right)z^{\beta-1}(\tau)\,d\tau\,dt = H(s) < \infty.$$

It follows from (8) that

$$\int_0^\infty \phi(\tau)z^{\beta-3}(\tau)z'^2(\tau)\,d\tau < \infty \tag{9}$$

since the integrand is nonnegative.

In view of all these, (6) implies

$$K - \left(\phi(s)z^{\beta-1}(s)\right)' + H(s) + (\beta - 1)G(s) = -\frac{\beta - 1}{\beta}\mathcal{Q}(s)$$

and so

$$\mathcal{Q}(s) \leqslant \frac{\beta}{\beta - 1}\left(\phi(s)z^{\beta-1}(s)\right)',$$

from which

$$\begin{aligned} \frac{\mathcal{Q}_+^2(s)}{s} &\leqslant \left(\frac{\beta}{\beta-1}\right)^2 \frac{\left(\phi(s)z^{\beta-1}(s)\right)'^2}{s} \\ &\leqslant 2\left(\frac{\beta}{\beta-1}\right)^2\left[\frac{\phi'^2(s)z^{2\beta-2}}{s} + \frac{(\beta-1)^2\phi^2(s)z^{2\beta-4}(s)z'^2(s)}{s}\right]. \end{aligned} \tag{10}$$

By (7) and (9),

$$\int_0^\infty \frac{\phi^2(s)z^{2\beta-4}(s)z'^2(s)}{s}\,ds \leqslant K_0\int_0^\infty \phi(s)z^{2\beta-3}(s)z'^2(s)\,ds < \infty, \tag{11}$$

where $K_0 = \sup_{t>0}\phi(t)z^{\beta-1}(t)/t$. By (7) again

$$\int_0^T \frac{\phi'^2(s)z^{2\beta-2}}{s}\,ds \leqslant K_0\int_0^T \frac{\phi'^2(s)s}{\phi^2(s)}\,ds. \tag{12}$$

Inequalities (10), (11) and (12) together contradict our hypothesis (4). This completes the proof of the theorem.

REMARK. For the case $a(t) = t^\lambda \sin t$ and $1 > \lambda > -\gamma$, we can choose $\phi(t) = t^\mu$ with any μ such that $0 < \mu < 1$ and $1 > \mu\gamma + \lambda > 0$. Denote $\mu\gamma + \lambda$ by θ. Then $\mathcal{Q}$ is defined and $\mathcal{Q}(s) = s^\theta(\cos s + o(1))$. Since $\phi'^2(s)s/\phi^2(s) = \mu^2/s$, (4) is satisfied and so (1) is oscillatory. The same argument works for $a(t) = t^\lambda f(t)$ with $\int_0^T f(t)\,dt = 0$ and $\lambda > -\gamma$.

The following result extends the necessity part of Belohorec's Theorem, i.e. equation (1) has a nonoscillatory solution if $a(t)$ satisfies

$$a(t) \geqslant 0, \qquad \int_0^\infty t^\gamma a(t)\,dt < \infty. \tag{13}$$

Condition (13) implies in particular that $\lim_{T\to\infty}\int_0^T a(t)\,dt$ exists and is finite when $a(t)$ is nonnegative.

THEOREM 2. *Suppose that* $A(t) = \int_t^\infty a(t)\,dt$ *exists for all* $t \geqslant 0$. *If there exists a function* $F(t) \in C^1[0,\infty)$ *such that* $|A(t)| \leqslant F(t)$ *for all large* t *where* $F(t) = O(t^{-\gamma})$ *as* $t \to \infty$ *and*

$$\int_0^\infty t^\gamma |F'(t)|\,dt = B_0 < \infty \tag{14}$$

then (1) *has a nonoscillatory solution.*

PROOF. Let $y_m(t)$ be the solution of (1) satisfying $y_m(1) = 0$, $y_m'(1) = m$, where m is a positive number. We claim that when m is large enough, $y_m'(t) > 0$ for all $t > 1$ and so y is nonoscillatory. For the sake of brevity, we omit the subscript m in the following discussion.

Suppose now that $y'(t) = 0$ for some $t > 1$. Let τ_1 be the smallest of such t. Let τ_2 be the smallest of all those t such that $y'(t) = 2m$. (If no such t exists, let $\tau_2 = \infty$.) Finally let $\tau = \min\{\tau_1, \tau_2\}$. Then on $[1, \tau)$, $0 < y'(t) < 2m$. It follows that

$$(15) \qquad 0 < y(t) < 2mt, \qquad t \in [1, \tau].$$

At $t = \tau$, we have either

$$(16) \qquad y'(\tau) = 0 \quad (\text{if } \tau = \tau_1) \qquad \text{or} \qquad y'(\tau) = 2m \quad (\text{if } \tau = \tau_2).$$

Integrating (1) once we have for $t \in [1, \tau]$

$$(17) \qquad y'(t) = m - \int_1^t a(s) y^\gamma(s)\, ds.$$

We now proceed to estimate the integral in (17) above as follows:

$$(18) \qquad \begin{aligned} \left|\int_1^t a(s) y^\gamma(s)\, ds\right| &= \left|(A(1) - A(t)) y^\gamma(t) + \int_1^t (A(s) - A(1)(y^\gamma - (s)))'\, ds\right| \\ &\leq y^\gamma(t)\{2|A(1)| + |A(t)|\} + \int_1^t |A(s)| (y^\gamma(s))'\, ds. \end{aligned}$$

(The last step uses the fact that $y(t), y'(t) > 0$ on $[1, \tau)$.) We now integrate the last integral in (18) above:

$$(19) \qquad \left|\int_1^t A(s)(y^\gamma(s))'\, ds\right| \leq \int_1^t F(s)(y^\gamma(s))'\, ds \leq F(t) y^\gamma(t) + \int_1^t |F'(s)| y^\gamma(s)\, ds.$$

Since $A(T)$ tends to zero as $T \to \infty$ by its very definition, $A(t)$ is bounded on $[1, \tau)$, say by a constant K. By assumption, there exists a constant B_1 such that $|t^\gamma F(t)| \leq B_1$. For $t \in [1, \tau)$, we also have from (15),

$$(20) \qquad \int_1^t |F'(s)| y^\gamma(s)\, ds \leq (2m)^\gamma \int_1^t |F'(s)| s^\gamma\, ds \leq B_0 (2m)^\gamma.$$

Using (19) and (20) in (18), we find

$$(21) \qquad \left|\int_1^t a(s) y^\gamma(s)\, ds\right| \leq (3K + B_0 + B_1)(2m)^\gamma = M(2m)^\gamma.$$

Substituting estimate (21) into (17), we obtain

$$m - (2m)^\gamma M \leq y'(t) \leq m + (2m)^\gamma M, \quad \text{for all } t \in [1, \tau].$$

For $m > (2^\gamma M)^{1/(1-\gamma)}$, we have in particular $0 < y'(\tau) < 2m$. This contradicts (16).

REMARK. For $\lambda < -\gamma$ and $a(\tau) = t^\lambda \sin t$, we see that $|\int_t^\infty a(s)\, ds|$ is less than a constant multiple of t^λ. Then $F(t) \equiv ct^\lambda$ satisfies the hypotheses of the theorem and so (1) is nonoscillatory.

Another example is offered by $a(t) = t^{-\gamma}(\log t)^\mu \sin t$, $\mu \leq -2$. We see that $F(t)$ can be taken to be a multiple of $t^{-\gamma}(\log t)^\mu$.

If F is any C^1 nondecreasing function such that

$$(22) \qquad \int_1^\infty \frac{F(t)}{t^{1-\gamma}}\, dt < \infty$$

then F satisfies the hypotheses of the theorem, that is $F(t) = O(t^{-\gamma})$, and (14) holds. To see this we apply integration by parts to obtain

$$\frac{1}{\gamma}F(T)T^{\gamma} + \int_1^T \frac{[-F'(t)]t^{\gamma}}{\gamma}\,dt = \int_1^T \frac{F(t)}{t^{1-\gamma}}\,dt + \frac{1}{\gamma}F(1).$$

Since the right-hand side is bounded, by (22), each of the terms on the left is bounded for all T. It can be shown by a continuity argument that the theorem still holds if F satisfies (22) but no continuity requirement is assumed on F.

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