ON THE OSCILLATION AND NONOSCILLATION OF
SECOND ORDER SUBLINEAR EQUATIONS

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Abstract. An oscillation criterion and a nonoscillation criterion are given for the
sublinear equation \( y'' + a(t) |y|^\gamma \text{sgn } y = 0, \quad 0 < \gamma < 1, \quad t \in (0, \infty), \) where \( a(t) \) is
allowed to change sign. When applied to the special case \( a(t) = t^\lambda \sin t, \) we deduce
oscillation for \( \lambda > -\gamma \) and nonoscillation for \( \lambda < -\gamma. \)

We are interested in determining when all continuable solutions of the sublinear
Emden-Fowler equation
\[
y''(t) + a(t) |y(t)|^\gamma y(t) = 0, \quad t \in [0, \infty), \quad 0 < \gamma < 1,
\]
are oscillatory. We are especially motivated by the particular case \( a(t) = t^\lambda \sin t \) or
more generally \( t^{\lambda} f(t) \) where \( f \) is a periodic function of period \( T \) such that \( \int_0^T f(t) \, dt > 0. \)

We shall use as weight functions those \( \phi : [0, \infty) \to [0, \infty) \) such that
\[
\phi' > 0, \quad \phi'' < 0.
\]

In an earlier paper [2], the authors proved the following extension of the
well-known Belohorec Theorem.

Theorem. If there exists a function \( \phi \) satisfying (2) such that
\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \int_0^\tau \phi'(\tau) a(\tau) \, d\tau \, dt = \infty,
\]
then (1) is oscillatory, i.e. all continuable solutions of (1) are oscillatory.

An immediate consequence of this theorem is that when \( a(t) = t^\lambda f(t) \) with
\( \int_0^T f(t) \, dt > 0 \) and \( \lambda \geq -\gamma \) or with \( \int_0^T f(t) \, dt = 0 \) and \( \lambda > 1 - \gamma, \) then (1) is oscilla-
tory. When \( \int_0^T f(t) \, dt = 0 \) and \( \lambda \leq 1 - \gamma, \) the theorem fails to apply.

The first result of this paper is a sufficient oscillation condition that applies to
cases in which
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^\tau \phi'(\tau) a(\tau) \, d\tau \, dt
\]
exists and is finite. This condition allows us to deduce oscillation for \( \lambda > -\gamma \) in our
motivating example. To supplement our first result, we prove a nonoscillation
theorem which allows us to settle the cases \( \lambda < -\gamma \). The critical case \( \lambda = -\gamma \) remains, however, unanswered. Thus the conjecture made by Butler in \([1, \text{p. 144}]\) is almost completely proved. For reference to other known results consult \([2]\) or \([3]\).

We define the functions

\[
\Omega(s) = \lim_{T \to \infty} \frac{1}{T} \int_s^T \int_s^T \phi^*(\tau) a(\tau) \, d\tau \, dt
\]

and

\[
\Omega_+ (s) = \max \{ \Omega(s), 0 \}.
\]

**Theorem 1.** If there is a weight function \( \phi \) satisfying \((2)\) so that the function \( \Omega \) given by \((3)\) is defined and satisfies

\[
\limsup_{T \to \infty} \left( \int_0^T \frac{A_+^2(s)}{s} \, ds \right) \left( \int_0^T \frac{\phi^2(s) s}{\phi^2(s)} \, ds \right)^{-1} = \infty,
\]

then \((1)\) is oscillatory.

**Proof.** As in \([2]\), the following easily verified identity plays a crucial role:

\[
(\phi z^\beta - 1)'' + (\beta - 1)\phi z^\beta z'^2 + \left( \frac{\phi''}{\beta} \right) z^\beta - 1 = - \left( \frac{\beta - 1}{\beta} \right) \phi' a,
\]

where \( z = (y/\phi)^\gamma \) and \( \beta = 1/\gamma > 1 \). Integrating \((5)\) twice, first over \([s, t]\), then over \([s, T]\), we obtain

\[
\phi(T) z^\beta(T) - \phi(s) z^\beta(s) - (\phi(s) z^\beta(s))' (T - s) \\
+ \int_s^T \int_s^T \left( \frac{\phi''}{\beta} \right) z^\beta - 1 (\tau) \, d\tau \, dt \\
+ (\beta - 1) \int_s^T \int_s^T \phi(\tau) z^\beta - 3 (\tau) z'^2 (\tau) \, d\tau \, dt \\
= - \left( \frac{\beta - 1}{\beta} \right) \int_s^T \int_s^T \phi'(\tau) a(\tau) \, d\tau \, dt.
\]

Dividing by \( T \) and letting \( T \to \infty \), we see that, because the right-hand side tends to a limit and the integrands of the two integrals on the left-hand side as well as the first term are nonnegative, the following limits exist and are finite:

\[
0 \leq \lim_{T \to \infty} \frac{\phi(T) z^\beta - 1 (T)}{T} = K < \infty,
\]

\[
0 \leq \lim_{T \to \infty} \frac{1}{T} \int_s^T \int_s^T \phi(\tau) z^\beta - 3 (\tau) z'^2 (\tau) \, d\tau \, dt = G(s) < \infty
\]

and

\[
0 \leq \lim_{T \to \infty} \frac{1}{T} \int_s^T \int_s^T \left( \frac{\phi''}{\beta} \right) z^\beta - 1 (\tau) \, d\tau \, dt = H(s) < \infty.
\]

It follows from \((8)\) that

\[
\int_0^\infty \phi(\tau) z^\beta - 3 (\tau) z'^2 (\tau) \, d\tau < \infty
\]

since the integrand is nonnegative.
In view of all these, (6) implies
\[ K - (\phi(s)z^{\beta-1}(s))^\prime + H(s) + (\beta - 1)G(s) = -\frac{\beta - 1}{\beta} \mathcal{Q}(s) \]
and so
\[ \mathcal{Q}(s) \leq \frac{\beta}{\beta - 1} (\phi(s)z^{\beta-1}(s))^\prime, \]
from which
\[ \frac{\mathcal{Q}^2(s)}{s} \leq \left( \frac{\beta}{\beta - 1} \right)^2 \frac{(\phi(s)z^{\beta-1}(s))^2}{s} \]
(10)
\[ \leq 2\left( \frac{\beta}{\beta - 1} \right)^2 \left[ \frac{\phi^2(s)z^{2\beta-2}}{s} + \frac{(\beta - 1)^2\phi^2(s)z^{2\beta-4}(s)z^2(s)}{s} \right]. \]
By (7) and (9),
\[ \int_0^\infty \frac{\phi^2(s)z^{2\beta-4}(s)z^2(s)}{s} ds \leq K_0 \int_0^\infty \frac{\phi(s)z^{2\beta-3}(s)z^2(s)}{s} ds < \infty, \]
where \( K_0 = \sup_{t \geq 0} \phi(t)z^{\beta-1}(t)/t. \) By (7) again
\[ \int_0^T \frac{\phi^2(s)z^{2\beta-2}}{s} ds \leq K_0 \int_0^T \frac{\phi^2(s)z^2(s)}{s} ds. \]
Inequalities (10), (11) and (12) together contradict our hypothesis (4). This completes the proof of the theorem.

Remark. For the case \( a(t) = t^\lambda \sin t \) and \( 1 > \lambda > -\gamma, \) we can choose \( \phi(t) = t^\mu \) with any \( \mu \) such that \( 0 < \mu < 1 \) and \( 1 > \mu \gamma + \lambda > 0. \) Denote \( \mu \gamma + \lambda \) by \( \theta. \) Then \( \mathcal{Q} \) is defined and \( \mathcal{Q}(s) = s^\theta (\cos s + o(1)). \) Since \( \phi^2(s)s/\phi^2(s) = \mu^2/s, \) (4) is satisfied and so (1) is oscillatory. The same argument works for \( a(t) = t^\lambda f(t) \) with \( \int_0^T f(t) dt = 0 \) and \( \lambda > -\gamma. \)

The following result extends the necessity part of Belohorec’s Theorem, i.e. equation (1) has a nonoscillatory solution if \( a(t) \) satisfies
\[ a(t) > 0, \quad \int_0^\infty \gamma a(t) dt < \infty. \]
Condition (13) implies in particular that \( \lim_{T \to \infty} \int_0^T a(t) dt \) exists and is finite when \( a(t) \) is nonnegative.

Theorem 2. Suppose that \( A(t) = \int_0^\infty a(t) dt \) exists for all \( t \geq 0. \) If there exists a function \( F(t) \in C^1[0, \infty) \) such that \( |A(t)| \leq F(t) \) for all large \( t \) where \( F(t) = O(t^{-\gamma}) \) as \( t \to \infty \) and
\[ \int_0^\infty \gamma |F'(t)| dt = B_0 < \infty \]
then (1) has a nonoscillatory solution.

Proof. Let \( y_m(t) \) be the solution of (1) satisfying \( y_m(1) = 0, y_m'(1) = m, \) where \( m \) is a positive number. We claim that when \( m \) is large enough, \( y_m'(t) > 0 \) for all \( t > 1 \) and so \( y \) is nonoscillatory. For the sake of brevity, we omit the subscript \( m \) in the following discussion.
Suppose now that \( y'(t) = 0 \) for some \( t > 1 \). Let \( \tau_1 \) be the smallest of such \( t \). Let \( \tau_2 \) be the smallest of all those \( t \) such that \( y'(t) = 2m \). (If no such \( t \) exists, let \( \tau_2 = \infty \).) Finally let \( \tau = \min(\tau_1, \tau_2) \). Then on \([1, \tau)\), \(0 < y'(t) < 2m\). It follows that
\[
0 < y(t) < 2mt, \quad t \in [1, \tau].
\]

At \( t = \tau \), we have either
\[
y'(\tau) = 0 \quad \text{(if \( \tau = \tau_1 \)) or} \quad y'(\tau) = 2m \quad \text{(if \( \tau = \tau_2 \)).}
\]

Integrating (1) once we have for \( t \in [1, \tau] \)
\[
y'(t) = m - \int_1^t a(s) y'(s) \, ds.
\]

We now proceed to estimate the integral in (17) above as follows:
\[
\left| \int_1^t a(s) y'(s) \, ds \right| = \left| (A(1) - A(t)) y'(t) + \int_1^t (A(s) - A(1)) (y'(s) - (s)) \right| \, ds \leq y'(t) \{2 |A(1)| + |A(t)| \} + \int_1^t |A(s)| (y'(s))' \, ds.
\]

(The last step uses the fact that \( y(t), y'(t) > 0 \) on \([1, \tau)\).) We now integrate the last integral in (18) above:
\[
\int_1^t A(s) (y'(s)') \, ds \leq \int_1^t F(s) (y'(s))' \, ds \leq F(t) y'(t) + \int_1^t F'(s) |y'(s)| \, ds.
\]

Since \( A(T) \) tends to zero as \( T \to \infty \) by its very definition, \( A(t) \) is bounded on \([1, \tau)\), say by a constant \( K \). By assumption, there exists a constant \( B_1 \) such that \( |r^\gamma F(t)| \leq B_1 \). For \( t \in [1, \tau) \), we also have from (15),
\[
\int_1^t |F'(s)| y'(s) \, ds \leq (2m)^\gamma \int_1^t |F'(s)| s^\gamma \, ds \leq B_0 (2m)^\gamma.
\]

Using (19) and (20) in (18), we find
\[
\left| \int_1^t a(s) y'(s) \, ds \right| \leq (3K + B_0 + B_1)(2m)^\gamma = M(2m)^\gamma.
\]

Substituting estimate (21) into (17), we obtain
\[
m - (2m)^\gamma M < y'(t) < m + (2m)^\gamma M, \quad \text{for all} \quad t \in [1, \tau].
\]

For \( m > (2M)^{1/(1-\gamma)} \), we have in particular \( 0 < y'(\tau) < 2m \). This contradicts (16).

Remark. For \( \lambda < -\gamma \) and \( a(t) = t^\lambda \sin t \), we see that \( \int_1^\infty a(s) \, ds \) is less than a constant multiple of \( t^\lambda \). Then \( F(t) = ct^\lambda \) satisfies the hypotheses of the theorem and so (1) is nonoscillatory.

Another example is offered by \( a(t) = t^{-\gamma}(\log t)^\mu \sin t \), \( \mu \leq -2 \). We see that \( F(t) \) can be taken to be a multiple of \( t^{-\gamma}(\log t)^\mu \).

If \( F \) is any \( C^1 \) nondecreasing function such that
\[
\int_1^\infty \frac{F(t)}{t^{1-\gamma}} \, dt < \infty
\]
then $F$ satisfies the hypotheses of the theorem, that is $F(t) = O(t^{-\gamma})$, and (14) holds. To see this we apply integration by parts to obtain

$$\frac{1}{\gamma} F(T) T^\gamma + \int_1^T \frac{[-F'(t)] t^\gamma}{\gamma} \, dt = \int_1^T \frac{F(t)}{t^{1-\gamma}} \, dt + \frac{1}{\gamma} F(1).$$

Since the right-hand side is bounded, by (22), each of the terms on the left is bounded for all $T$. It can be shown by a continuity argument that the theorem still holds if $F$ satisfies (22) but no continuity requirement is assumed on $F$.

**References**

