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ON FOURIER INTEGRAL OPERATORS

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ABSTRACT. We consider operators of the form: $\int_{-\infty}^{\infty} F_t \varphi(t)\,dt$, where F_t is a 1-parameter family of Fourier integral operators and $\varphi(t)\,dt$ a tempered distribution on the real line and show that these operators are sums of pseudodifferential and Fourier integral operators. Here, we give the typical case where $\varphi(t)\,dt = \text{p.v.}\{1/t\}$. An application to singular integrals on variable curves is given.

THEOREM. *Let F_t be a 1-parameter family of local Fourier integral operators such that the kernel of F_t has an integral representation of the form: $\int_{\mathbf{R}^2} e^{i\phi(x,y,t,\theta)} a(x,y,t,\theta)\,d\theta$ with a and ϕ depending smoothly on t, a (resp. ϕ) a symbol in the class $S^0_{1,0}$ i.e.: $|\partial_x^\alpha \partial_y^\beta \partial_t^k \partial_\theta^\gamma a| \leq C_{\alpha,\beta,k,\gamma}(1+|\theta|)^{-|\gamma|}$ (resp. nondegenerate phase function) and $a(x,y,0,\theta) = 1$ (resp. $\phi(x,y,0,\theta) = (x-y)\cdot\theta$). Furthermore, we assume that $\dot{\phi}_\theta \times \ddot{\phi}_\theta \neq 0$; then the operator $T = \int_{-1}^{1} F_t\,dt/t$ can be written in the form $T = P + F$ where P is a ψ.d.o. with symbol in $S^0_{1/2,1/2}$ and F a Fourier integral operator with amplitude in $S^{-1/2}_{1,0}$.*

The proof follows from the asymptotic expansion of the function

$$\omega(x,y,\theta) = B(\theta)\int_{-1}^{1} e^{i\phi(x,y,t,\theta)} a(x,y,t,\theta)\frac{dt}{t}$$

where B is a C^∞ function on $\mathbf{R}^2$, identically zero near the origin and homogeneous of degree zero for all $|\theta| \geq 1$. For simplicity, we will not, in the sequel, make mention of B.

We, now, use Taylor expansion to have:

$$\phi(x,y,t,\theta) = (x-y)\cdot\theta + tv(x,y,\theta) + t^2 w(x,y,t,\theta)/2$$

where $v = v(x,y,\theta) = \dot{\phi}(x,y,0,\theta)$. We will also write $w = w(x,y,\theta) = \ddot{\phi}(x,y,0,\theta)$.

From the hypothesis, we have $v_\theta \times w_\theta \neq 0$; hence using the homogeneity of v and w, for any (x_0,y_0,θ_0), there exists a conic neighborhood $K \times \Gamma$ of (x_0,y_0,θ_0) on which either $|v(x,y,\theta)| \geq a|\theta|$ or $|w(x,y,\theta)| \geq b|\theta|$ where a,b are some positive constants depending only on $K \times \Gamma$. We consider separately the two types of cones. We let

$$\Gamma_1 = \{\theta \in \mathbf{R}^2 \colon v(x,y,\theta) \geq a|\theta| \text{ for all } (x,y) \in K\},$$
$$\Gamma_2 = \{\theta \in \mathbf{R}^2 \colon w(x,y,\theta) \geq b|\theta| \text{ for all } (x,y) \in K\}.$$

The two other cones can be considered in a similar fashion.

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We first consider Γ_2,

LEMMA 1. *The symbol*

$$\sigma_1 = \int_{|t|<1/\sqrt{w}} \exp\left\{itv + \frac{t^2}{2} w(x,y,t,\theta)\right\} a(x,y,t,\theta)\frac{dt}{t}$$

belongs to $S^0_{1/2,1/2}(K \times \Gamma_2)$.

PROOF. We write

$$\sigma_1 = \int_{|t|\leq 1/\sqrt{w}} \exp\left\{itv + \frac{t^2 w}{2}\right\}\frac{dt}{t} + E_1$$

where

$$E_1 = \int_{|t|\leq 1/\sqrt{w}} \exp\left\{itv + \frac{t^2}{2}w\right\}\left(\exp\left\{i\frac{t^2}{2}(w(x,y,t,\theta)-w)\right\}a(x,y,t,\theta)-1\right)\frac{dt}{t}$$
$$= \int_{|t|\leq 1} \exp\left\{it\frac{v}{\sqrt{w}}\right\}\frac{1}{t}\left(\exp\left\{i\frac{t^2}{2}(w(t/\sqrt{w})/w-1)\right\}a\left(\frac{t}{\sqrt{w}}\right)-1\right)\exp\left\{i\frac{t^2}{2}\right\}dt$$

where $w(t) = w(x,y,t,\theta)$ and $a(t) = a(x,y,t,\theta)$. We then have

(1) $e^{itv/\sqrt{w}}$ belongs to $S^0_{1/2,1/2}(K \times \Gamma_2)$ uniformly in t, $|t| \leq 1$,

(2) $w(t/\sqrt{w})/w$ belongs to $S^0_{1,0}(K \times \Gamma_2)$ uniformly in t, $|t| \leq 1$. Also $\underline{a} = a(t/\sqrt{w})$ verifies:

$$|\partial_x^\alpha \partial_y^\beta \partial_t^k \partial_\theta^\gamma \underline{a}| \leq c_{\alpha,\beta,k,\gamma}(1+|\theta|)^{-(|\gamma|+k/2)}$$

and

$$\left|\frac{1}{t}\left(\exp\left\{i\frac{t^2}{2}(w(t/\sqrt{w})/w-1)\right\}a(t/\sqrt{w})-1\right)\right| \leq (\mathrm{const})(1+|\theta|)^{-1/2}$$

on $K \times \Gamma_2$. Thus $\frac{1}{t}(\exp\{i\frac{t^2}{2}(w(t/\sqrt{w})/w-1)\}a(t\sqrt{w})-1)$ and therefore E_1, verifies the conclusion of Lemma 1.

For $\sigma_1 - E_1$, we have

$$\sigma_1 - E_1 = \int_{|t|\leq 1} e^{it\frac{v}{\sqrt{w}}}\frac{e^{i\frac{t^2}{2}}}{t}\,dt = \int_{|t|\leq 1}\exp\left\{it\frac{v}{\sqrt{w}}\right\}\frac{dt}{t}$$
$$+ \int_{|t|\leq 1}\exp\left\{it\frac{v}{\sqrt{w}}\right\}\frac{\exp\left\{i\frac{t^2}{2}\right\}-1}{t}\,dt = \sigma_{11} + \sigma_{12}.$$

σ_{12} is easily seen to be in $S^0_{1/2,1/2}(K \times \Gamma_2)$ since $v/\sqrt{w}$ is in $S^{1/2}_{1/2,1/2}(K \times \Gamma_2)$ and $\frac{1}{t}(e^{it^2}-1)$ is a smooth function. For σ_{11}, we write

$$\sigma_{11} = \int_0^{v/\sqrt{w}} \frac{\sin u}{u}\,du = F\left(\frac{v}{\sqrt{w}}\right)$$

from which one deduces that σ_{11} belongs to $S^0_{1/2,1/2}(K \times \Gamma_2)$. This completes the proof of Lemma 1.

We now wish to estimate,

$$\sigma_2 = \int_{1/\sqrt{w}\le|t|\le 1} e^{i\phi(x,y,t,\theta)} a(x,y,t,\theta)\frac{dt}{t}.$$

We let $t(x,y,\theta)$ be such that $\dot\phi(x,y,t(x,y,\theta),\theta)=0$. From the hypothesis we have $\ddot\phi(x,y,t(x,y,\theta),\theta)\neq 0$. Also, $t(x,y,\theta)$ is clearly homogeneous of degree 0 in θ.

LEMMA 2. *The operator corresponding to σ_2 is a Fourier integral operator with phase function $\tilde\phi(x,y,\theta)=\phi(x,y,t(x,y,\theta),\theta)$ and amplitude in $S^{-1/2}_{1,0}(K\times\Gamma_2)$.*

PROOF. We use the stationary phase method to write the full expansion of σ_2 whose first term is

$$\frac{e^{i\pi/4}}{t(x,y,\theta)} e^{i\phi(x,y,t(x,y,\theta),\theta)} \frac{a(x,y,t(x,y,\theta),\theta)}{\sqrt{\ddot\phi(x,y,t(x,y,\theta),\theta)}}.$$

Clearly $\tilde\phi = \phi(x,y,t(x,y,\theta),\theta)$ is homogeneous of degree 1 in θ. Also from $\dot\phi(x,y,t(x,y,\theta),\theta)=0$, we see that

$$\tilde\phi_x = \phi_x(x,y,t,\theta)_{|t=t(x,y,\theta)}, \quad \tilde\phi_\theta = \phi_\theta(x,y,t,\theta)_{|t=t(x,y,\theta)},$$
$$\tilde\phi_{x\theta} = \phi_{x\theta}(x,y,t,\theta)_{|t=t(x,y,\theta)}$$

which shows that $\tilde\phi$ is nondegenerate since ϕ is. Also from $\dot\phi(x,y,t(x,y,\theta),\theta)=0$ we have $\dot\phi_x+\ddot\phi t_x=0$; hence $t_x=-\dot\phi/\ddot\phi$; in particular t_x is bounded on $K\times\Gamma_2$. Similarly for t_y and all higher derivatives in x and y of $t(x,y,\theta)$. Now, it is clear that

$$\frac{1}{t(x,y,\theta)}\frac{a(x,y,t(x,y,\theta),\theta)}{\sqrt{\ddot\phi(x,y,t(x,y,\theta),\theta)}}$$

belongs to $S^{-1/2}_{1,0}(K\times\Gamma_2)$. The expansion given by the stationary phase method gives an asymptotic expansion of the amplitude (use formula 2.14, p. 431, [2] with $\rho=1$ $f(t)=\phi(x,y,t,\phi)$ and $g(t)=a(t)/t$ and the fact that $t(x,y,\theta)$ stays away from 0 on $K\times\Gamma_2$).

We now consider Γ_1,

LEMMA 3. *The symbol*

$$\nu_1 = \int_{|t|\le 1/\sqrt{v}} \exp\left\{itv+\frac{t^2}{2}w(t)\right\}a(t)\frac{dt}{t}$$

belongs to $S^0_{1/2,1/2}(K\times\Gamma_1)$.

PROOF. We write

$$\nu_1 = \int_{|t|\le 1/\sqrt{v}} \exp\left\{itv+\frac{t^2}{2}w\right\}\frac{dt}{t}+F_1$$

where

$$F_1 = \int_{|t|\le 1/\sqrt{v}} \exp\left\{itv+\frac{t^2}{2}w\right\}\left(\exp\left\{i\frac{t^2}{2}(w(x,y,t,\theta)-w)\right\}a(x,y,t,\theta)-1\right)\frac{dt}{t}$$
$$= \frac{1}{\sqrt{v}}\int_{|t|\le 1} e^{it\sqrt{v}}\frac{1}{t/\sqrt{v}}\left(\exp\left\{i\frac{t^2}{2}\left(w\left(x,y,\frac{t}{\sqrt{v}},\theta\right)-w\right)/v\right\}\right.$$
$$\left.\cdot\, a\left(x,y\frac{t}{\sqrt{v}},\theta\right)-1\right)e^{it^2/2}\,dt$$

which is easily seen to be in $S^{-1/2}_{1,0}(K \times \Gamma_1)$ (see similar argument for E_1).
For $\nu_1 - F_1$, we have

$$\nu_1 - F_1 = \int_{|t|\leq 1} \exp\left\{it\sqrt{v} + \frac{t^2}{2}\frac{w}{v}\right\}\frac{dt}{t}$$
$$= \int_{|t|\leq 1} e^{it\sqrt{v}}\frac{dt}{t} + \frac{w}{v}\int_{|t|\leq 1} e^{it\sqrt{v}}\frac{e^{it^2w/v} - 1}{t^2\frac{w}{v}}\, t\, dt.$$

Both terms are clearly in $S^0_{1/2,1/2}(K \times \Gamma_1)$ which finishes the proof of Lemma 3.

We now let $\psi_0(u)$ be a smooth function, $\psi_0(-u) = \psi_0(u)$, $\psi_0 \equiv 1$ for $-\frac{1}{2} \leq u \leq \frac{1}{2}$ and $\psi_0 \equiv 0$ for $|u| \geq 1$ and put $\psi_1 = 1 - \psi_0$. The conclusion of Lemma 3 is clearly also valid for

$$\nu_1' = \int_{-1}^{1} \exp\left\{itv + \frac{t^2}{2}w(x,y,t,\theta)\right\}a(x,y,t,\theta)\psi_0(t\sqrt{v})\frac{dt}{t}.$$

LEMMA 4. *The operator corresponding to*

$$\nu_2' = \int_{-1}^{1} \exp\left\{itv + \frac{t^2}{2}w(x,y,t,\theta)\right\}a(x,y,t,\theta)\psi_1(t\sqrt{v})\frac{dt}{t}$$

is a Fourier integral operator with phase function equal to $\phi(x,y,\pm 1,\theta)$ and amplitude in $S^{-1}_{1,0}(K \times \Gamma_1)$.

PROOF. We write

$$\nu_2' = \int_{-\sqrt{v}}^{\sqrt{v}} e^{it\sqrt{v}} \exp\left\{i\frac{t}{2}w\left(x,y,\frac{t}{\sqrt{v}},\theta\right)/v\right\}a\left(x,y,\frac{t}{\sqrt{v}},\theta\right)\psi_1(t)\frac{dt}{t}$$
$$= \int_{-\sqrt{v}}^{-1} + \int_{1}^{\sqrt{v}}.$$

Now, an integration by parts gives

$$\int_1^{\sqrt{v}} = \frac{1}{v}e^{iv+\frac{1}{2}w(x,y,1,\theta)}a(x,y,1,\theta)\psi_1(\sqrt{v})$$
$$- \frac{1}{\sqrt{v}}\int_1^{\sqrt{v}} e^{it\sqrt{v}}\frac{d}{dt}\left(\exp\left\{i\frac{t^2}{2}w\left(x,y,\frac{t}{\sqrt{v}},\theta\right)/v\right\}a\left(x,y,\frac{t}{\sqrt{v}},\theta\right)\psi_1(t)\frac{1}{t}\right)dt.$$

Since $e^{iv+w(x,y,1,\theta)/2} = e^{i\phi(x,y,1,\theta)-i(x-y)\theta}$, the first term verifies clearly the conclusion of Lemma 4. Also, in the second term, a repeated integration by parts gives an asymptotic expansion of the amplitude. We have the same conclusion for the integral $\int_{-\sqrt{v}}^{-1}$. This finishes the proof of Lemma 4.

The proof of our result follows from the above lemmas and the use of a microlocal partition of unity associated with the different cones and a neighborhood of the origin. Notice that

$$E(x,y,\theta) = (1 - B(\theta))\int_{-1}^{1}(e^{i\phi(x,y,t,\theta)}a(x,y,t,\theta))\frac{dt}{t}$$
$$= (1 - B(\theta))\int_{-1}^{1}(e^{i\phi(x,y,t,\theta)}a(x,y,t,\theta) - e^{i(x-y)\cdot\theta})\frac{dt}{t}$$

is a symbol of a smoothing operator.

We have the following:

COROLLARY. *Let $\gamma(x,t)$ be a variable curve of class C^∞ in $\mathbf{R}^2$ such that $\gamma(x,0) = 0$ and $\dot\gamma \times \ddot\gamma \neq 0$ then the operator*

$$\mathcal{H} f(x) = \text{p.v.} \int_{-1}^{1} f(x - \gamma(x,t)) \frac{dt}{t}$$

is a bounded operator from $L^2_{\text{comp}}(\mathbf{R}^2)$ to $L^2_{\text{loc}}(\mathbf{R}^2)$ (see [3]).

PROOF. We write

$$f(x - \gamma(x,t)) = \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} e^{i[(x-y)\theta - \gamma(x,t)\cdot\theta]} f(y)\, dy d\theta$$

and apply our result with $a \equiv 1$ and $\phi(x,y,t,\theta) = (x-y)\theta - \gamma(x,t)\theta$. In particular there exists a ψ.d.o. P with symbol in $S^0_{1/2,1/2}$ such that $(\mathcal{H} - P)^*(\mathcal{H} - P)$ is a ψ.d.o. with symbol in $S^{-1}_{1,0}$.

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