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A REMARK ON EXPANDING MAPS

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ABSTRACT. In this paper we discuss the following problem stated by L. Nirenberg: Let T be an expanding map $H \to H$ (H is a Hilbert space) with $T(0) = 0$. Suppose T maps a neighborhood of the origin onto a neighborhood of the origin. Does T map H onto H?

We answer positively the problem when T is differentiable.

In [1] L. Nirenberg stated the following open problem:

Suppose T is a continuous map $H \to H$ (H is a Hilbert space) which is expanding, i.e. $\|Tx - Ty\| \geq \|x - y\|$, and $T(0) = 0$. Suppose T maps a neighborhood of the origin onto a neighborhood of the origin. Does T maps H onto H? If we consider αT instead of T for a real $\alpha > 1$, with no loss of generality, we may assume that $\|Tx - Ty\| \geq \alpha\|x - y\|$.

When H is a Euclidean space R^n, we know that it is true because of the Domain Invariance Theorem (in this case the condition that T maps a neighborhood of the origin onto a neighborhood of the origin can be omitted). When H is a Hilbert space there is a Domain Invariance Theorem for the following cases:

(i) $T = I - C$ where I is the identity and C is a compact operator, or

(ii) T is a strongly monotone operator.

(In case (i) see [2], in case (ii) see [3].) But, in general, the Domain Invariance Theorem does not hold; a simple counterexample is when T is the shift operator.

In this paper we answer positively the problem when T is differentiable.

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LEMMA 1. *Suppose T is an expanding map from a Banach space X into a Banach space Y, $T(0) = 0$, T maps a neighborhood $N_x(0)$ of the origin of X onto a neighborhood $N_Y(0)$ of the origin of Y, and T is Fréchet-differentiable at the origin of X, then $[T'(0)]^{-1}$ exists and $\|[T'(0)]^{-1}\| \leq 1/\alpha < 1$.*

PROOF. First we prove $RT'(0)$ (range of $T'(0)$) is dense in Y. If the statement of the lemma is not true, there must exist a $z_0 \in Y^*$ (Y^* is the conjugate space) such that $z_0(T'(0)x) = 0\ \forall x \in X$.

Set

$$T(x) = T(0) + T'(0)x + 0(\|x\|),$$

we have

$$z_0(T(x)) = z_0(T(0) + T'(0)x) + z_0(0(\|x\|)).$$

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There exists a $\bar{z}_0 \in Y$ such that $z_0(\bar{z}_0) = ||\bar{z}_0||||z_0||_*$. We know that T maps $N_x(0)$ onto $N_Y(0)$ from assumption, so that there exists $\{x_n\} \subset N_x(0)$ such that $Tx_n = \alpha_n \bar{z}_0$. By $||Tx_n|| > ||x_n||$ and $\alpha_n \to 0$ we find $x_n \to 0$ and

$$(1) \quad |z_0(T(x_n))| = |z_0(\alpha_n \bar{z}_0)| = ||\alpha_n \bar{z}_0||||z_0||_* = ||Tx_n||||z_0||_* \geq ||x_n||||z_0||_*.$$

On the other hand

$$(2) \quad |z_0(T(x_n))| = |z_0(T'(0)x_n) + z_0(0(||x_n||))| = |z_0(0(||x_n||))| \leq ||z_0||_* ||0(||x_n||)||.$$

By (1) and (2) we have

$$||x_n|| \leq ||0(||x_n||)|| \quad \text{as } x_n \to 0$$

This contradiction shows $RT'(0)$ is dense in Y.

Second we show $RT'(0) = Y$. It is sufficient to prove $RT'(0)$ is a closed set in Y.

Suppose $T'(0)x_n = y_n \to y$. We know that there exists the inverse of $T'(0)$ on $RT'(0)$. Since T is expansive therefore $||[T'(0)]^{-1}|| \leq 1/\alpha < 1$ on $RT'(0)$ and $x_n = [T'(0)]^{-1}y_n$, $||x_n - x_m|| \leq ||[T'(0)]^{-1}||||y_n - y_m|| \to 0$, so that there exists $x \in X$ such that $x_n \to x$ and $T'(0)x = y$, this fact show $[T'(0)]^{-1}$ exists and $||[T'(0)]^{-1}|| < 1$ on Y. Q.E.D.

The following Lemma 2 is an immediate corollary of the Theorem 2 in [4]. We only need the following special case in [4].

PROPOSITION 1. *Suppose X and Y are real Banach spaces, f is a map from X into Y, f has a linear Gâteaux differential $f'(x)$, a bounded linear operator, at every point $x \in X$, and $N([f'(x)]^*) = 0\ \forall x \in X$, where $N([f'(x)]^*)$ denotes the null space of $[f'(x)^*]$ and $[f'(x)]^*$ denotes the adjoint operator of $[f'(x)]$. If $f(X)$ is closed in Y then $f(X) = Y$.*

LEMMA 2. *Suppose X and Y are real Banach spaces, T is a map from X into Y with closed $T(X)$, T has a linear Gâteaux differential $T'(x)$, a bounded linear operator, at every point $x \in H$. If $[T'(x)]^{-1}$ exists $\forall x \in X$ then $T(X) = Y$.*

PROOF. By Proposition 1 it is sufficient to prove $N[T'(x)^*] = 0\ \forall x \in X$. If it is not true then there must exist a $x_0 \in X$ such that $N(T'(x_0)^*) \neq 0$, so that there exists $y_0^* \neq 0$, $y_0^* \in Y^*$ such that $[T'(x_0)]^* y_0^*(x) = 0$, i.e. $y_0^*(T'(x_0))(x) = 0$ but $RT'(x_0) = Y$ therefore we have $y_0^* = 0$ this is a contradiction. Q.E.D.

LEMMA 3. *Suppose T_1, T_2 are linear operators and T_1 has a bounded inverse T_1^{-1} with $||T_1^{-1}|| < 1$. If $||T_1 - T_2|| < 1$ holds, then T_2 has a bounded inverse T_2^{-1}.*

PROOF. We know $T_2 = T_1(I + (T_2 - T_1)T_1^{-1})$, since $||(T_2 - T_1)T_1^{-1}|| \leq ||T_2 - T_1||||T_1^{-1}|| < 1$, so that $[I + (T_2 - T_1)T_1^{-1}]^{-1}$ exists and T_2 has a bounded inverse. Q.E.D.

THEOREM 1. *Suppose X is a real Banach space and Y is a Banach space, also suppose T is a expanding map from X into Y, T is Fréchet-differentiable in X and $\forall x_0 \in X$ we have $\overline{\lim}_{x \to x_0} ||T'(x) - T'(x_0)|| < 1$, $T(0) = 0$ and T maps a neighborhood of the origin of X onto a neighborhood of the origin of Y. Then $TX = Y$.*

PROOF. Firstly, it is easy to show that the image of any expanding map is closed. If the statement of the theorem is not true then by Lemmas 2 and 3, we know that there exists a nonempty closed set $S \subset X$, such that $[T'(x)]$ does not have an inverse $\forall x \in S$. From Lemmas 1, 3 and the assumption of the theorem, we have a neighborhood of the origin, which does not intersect with S. Therefore there exists a ray intersecting with S, say $\{t\overline{x} \mid t \in \mathbf{R}'_+\}$. Then there is a $r > 0$ such that $t\overline{x} \in S$ for $t \in [0, r)$ and $r\overline{x} \in S$. Let $x_n = (r - 1/n)\overline{x}$, we have $x_n \notin S$ i.e. $[T'(x_n)]^{-1}\exists$, and we have $\|[T'(x_n)]^{-1}\| \leq 1/\alpha < 1$. Again, by assumption

$$\overline{\lim_{x_n \to r\overline{x}}} \|T'(x_n) - T'(r\overline{x})\| < 1,$$

and by Lemma 3 we know that $[T'(r\overline{x})]^{-1}$ exist, which contradicts $r\overline{x} \in S$. Q.E.D.

REMARK 1. In particular when $T \in C^1$ Theorem 1 is true, we could prove it by the open and closed mapping argument instead of Lemma 2.

REMARK 2. Suppose $X = Y = H$ is a Hilbert space, T is an expanding map and has a linear Gâteaux differential $T'(x)$ at every point $x \in H$, which is selfadjoint. Then T maps H onto H. We could find this conclusion from the fact that the residual spectrum of a selfadjoint operator is empty.

REMARK 3. In Theorem 1 if T satisfies

$$\|Tx - Ty\| \geq \alpha\|x - y\| \quad \forall x, y \in X,$$

where $\alpha > 0$ and if

$$\overline{\lim_{x \to x_0}} \|T'(x) - T'(x_0)\| < \alpha \quad \forall x_0 \in X.$$

Then $TX = Y$.

The following Proposition 2 is the Theorem 5.1 in [5].

PROPOSITION 2. *Suppose X is a Banach space, F is an operator from X into the conjugative space X^*, F has a linear Gâteaux differential $DF(x, h)$ at every point of the ball $B(x_0, r)$. The functional $(DF(X, h_1), h_2)$ is continuous in x at every point of $B(x_0, r)$, the operator F is potential in $B(x_0, r)$, then the bilinear functional $(Df(x, h_1), h_2)$ is symmetric for every $x \in B(x_0, r)$.*

By Proposition 2 we have the following result:

THEOREM 2. *Suppose T is an expanding map from H into H, $T \in C^1$ and T is a potential operator, then T maps H onto H.*

PROOF. For any $x_0 \in H$ there is a ball $B(x_0, r)$ such that in which the conditions in Proposition 2 are satisfied. Therefore $(DT(x, h_1), h_2)$ is symmetric for every $x \in B(x_0, r)$ and since T is Fréchet-differentiable we know that $T'(x)$ is a selfadjoint operator $\forall x \in H$. By Remark 2 we have $TH = H$. Q.E.D.

The referee has pointed out that Theorem 2 extends to a reflexive Banach space.

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