ON THE MONODROMY OF HIGHER LOGARITHMS

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Abstract. The (multivalued) higher logarithms are interpreted, by studying their monodromy, as giving well-defined maps from $\mathbb{P}^1 \setminus \{3 \text{ points}\}$ into certain complex nilmanifolds with $\mathbb{C}^*$-actions.

The purpose of this note is to exhibit a family of unipotent representations of $\mathbb{Z} \times \mathbb{Z}$ arising naturally from the monodromy of the higher logarithms $\ln_k$ (see [4]), and thereby interpret each $\ln_k$ as yielding a well-defined map $\rho_k$ of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ into a $(k+1)$-dimensional complex nilmanifold $M_{k+1}$ equipped with a $\mathbb{C}^*$-action. Moreover, a natural holomorphic connection $\nabla_k$ is shown to exist on each fibration $M_{k+1} \to M_{k+1}/\mathbb{C}^*$, with respect to which $\rho_k$ is flat. The role of dilogarithm in the study of volumes of hyperbolic 3-manifolds [5], arithmetic [1,2], $K$-theory and Kac-Moody Lie algebras [3] leads one to hope for such interesting links in the case of higher logarithms as well. Furthermore, the tower of nilmanifolds associated to $\mathbb{P}^1 \setminus \{3 \text{ points}\}$ via $\{\ln_k\}$ suggests, following a remark of P. Deligne, relations to Sullivan’s theory of differential forms. We hope to pursue this at some future time.

Finally, we have come to learn recently of an independent and very elegant construction of these nilmanifolds due to J. W. Milnor (unpublished).

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For $x$ in $\mathbb{C}$, set $\ln_0(x) = x/(1-x)$ and $L(x) = (2\pi i)^{-1}\log(x)$. Then $\ln_k$ is defined recursively by

$$\ln_k(x) = \int_0^x \ln_{k-1}(t) \, dL(t).$$

It follows easily that each $\ln_k$ satisfies the $(k+1)$st order, homogeneous, algebraic differential equation

$$(*) \quad \frac{d}{dx} \left( (1-x) \frac{d}{dx} \left( x \frac{d}{dx} U \right)^{k-1} \right) = 0.$$

This equation has regular singular points at 1 and $\infty$ if $k = 1$, and at 0, 1, and $\infty$ if $k > 1$. 

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Now let, for each integer $m > 2$,

$$N_m = \left\{ \begin{pmatrix} 1 & \cdots & \ast \\ \vdots \\ 0 & \cdots & 1 \end{pmatrix} \right\} \subset \text{GL}_m, \quad U_m = \left\{ \begin{pmatrix} 1 & \cdots & \ast \\ \vdots \\ 0 & I \end{pmatrix} \right\},$$

and let $[ ]_m$ denote the one-parameter subgroup $G_a \to N_m$ given by

$$y \mapsto \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & y & y^2/2! & y^3/3! & \cdots & y^{(m-1)}/(m-1)! \\ \vdots \\ 1 & \cdots \\ 0 & \cdots & y \end{pmatrix}.$$

The image of $[ ]_m$ normalizes $U_m$. Let $R_m$ denote the corresponding semidirect product. It is then an algebraic subgroup of $N_m$ isomorphic to $G_a \ltimes G_a^{m-1}$. We will write the elements of $R_m$ as $([y]_m; (x_1, \ldots, x_{m-1}))$ with $y, x_i$ in $G_a$. Let $R_2$ denote $G_a \times G_a$.

Let $\alpha$ and $\beta$ denote respectively the cycles around 0 and 1 in $P^1 \setminus \{0, 1, \infty\}$, oriented in the usual way. Then $\pi_1(P^1 \setminus \{0, 1, \infty\})$ is a free group on $\alpha$ and $\beta$. Now for each $k \geq 1$, define a representation $\lambda_k: \pi_1 \to \text{GL}_k+1(C) \subset \text{GL}_{k+1}(C)$ by

$$\alpha \mapsto ([1]_{k+1}; (0, \ldots, 0)) \quad \text{and} \quad \beta \mapsto ([0]_{k+1}; (1, 0, \ldots, 0)).$$

Put $\Gamma_{k+1} = \lambda_k(\pi_1)$. It is a discrete, $k$-step nilpotent subgroup of $\text{GL}_{k+1}(C)$. Note that $\Gamma_3$ is the $\mathbb{Z}$-points of the 3-dimensional Heisenberg group and that $\Gamma_2 = \mathbb{Z}^2 = \pi_1^{ab} = \pi_2^{ab}$, for every $k \geq 1$.

Let $M_{k+1}$ denote the complex nilmanifold $\Gamma_{k+1} \setminus \text{R}_{k+1}(C)$.

**Theorem.** (a) The (multivalued map) $p_k: P^1 \setminus \{0, 1, \infty\} \to R_{k+1}(C) \subset N_{k+1}$, given by

$$x \mapsto ([L(x)]_{k+1}; \ln_1(x), \ln_2(x), \ldots, \ln_k(x)),$$

becomes well defined modulo $\Gamma_{k+1}$.

(b) Each $M_{k+1}$ comes equipped with a $C^*$-action with the quotient being identified with $M_k$. If $p_k$ denotes the corresponding projection $M_{k+1} \to M_k$, then we have the commutative diagram (for $k \geq 2$)

$$\begin{array}{ccc}
P^1 \setminus \{0, 1, \infty\} & \xrightarrow{\rho_{k+1}} & M_{k+1} \\
\downarrow p_k & & \\
\rho_{k-1} & \xrightarrow{\rho_k} & M_k
\end{array}$$

(c) There exists a holomorphic connection $\nabla_k$ on $M_{k+1}$ such that $\rho_k$ yields a flat section to the corresponding pullback (via $\rho_{k-1}$) bundle with connection on $P^1 \setminus \{0, 1, \infty\}$.

**Proof.** (a) The solution space to $(\ast)_k$ is spanned by $\{\ln_k, L^j/j! | 0 \leq j \leq k-1\}$. It is easy to check that the monodromy around 1 amounts to sending $\ln_k$ to
\[ \ln_k + L^{(k-1)}/(k-1)! \] and fixing the \((L^j/j!)/j\)'s. Consequently, \(\beta\) acts on

\[
\begin{pmatrix}
\ln_k \\
L^{(k-1)}/(k-1)! \\
\vdots \\
L \\
1
\end{pmatrix}
\]

by multiplication on the left by \(\rho_k(\beta)\).

The monodromy around 0 fixes \(\ln_k\) and sends each \(L^j/j!\) to

\[
\sum_{0 \leq i \leq j} \frac{1}{(j-i)!} \cdot \frac{L^j}{i!}.
\]

Thus \(\alpha\) acts on \(v_k\) via \(\rho_k(\alpha)\) on the left. The assertion (a) now follows, since \(\lambda_k\) sends \(x\) to the matrix whose \((l+1)\)st column is

\[
\begin{pmatrix}
1 \\
l \\
0
\end{pmatrix}_l
\]

(b) Define \(p_k: R_{k+1} \to R_k\) when \(k \geq 2\) (resp. \(k = 2\)) by

\[
([y]_{k+1}; (x_1, \ldots, x_k)) \to ([y]_k; (x_1, \ldots, x_{k-1}))
\]

(resp. \(([y]; (x_1, x_2)) \to (y, x_1))

In either case let \(i_k\) denote the injection \(G_a \to R_{k+1}\) given by \(z \to ([0]_{k+1}; (0, \ldots, 0, z))\). Then we have the exact sequence

\[ 0 \to G_a \xrightarrow{i_k} R_{k+1} \xrightarrow{p_k} R_k \to 1. \]

Note that since \(\Gamma_{k+1}\) is a discrete, \(k\)-step nilpotent subgroup of \(R_{k+1}(C)\), the intersection of \(i_k(Q)\) with \(\Gamma_{k+1}\) is a rank-1 free abelian group, and is generated by \(i_k(t_k)\) for some \(t_k\) in \(Q\).

The map \(p_k\) makes sense on \(\Gamma_{k+1}\) and \(p_k(\Gamma_{k+1}) = \Gamma_k\). Consequently we have the commutative exact diagram

\[
\begin{array}{cccccc}
0 & \to & C & \xrightarrow{i_k} & R_{k+1}(C) & \xrightarrow{p_k} & R_k(C) & \to & 1 \\
\cup & & \cup & & \cup & & \cup & & \\
0 & \to & t_k Z & \xrightarrow{i_k} & \Gamma_{k+1} & \xrightarrow{p_k} & \Gamma_k & \to & 1.
\end{array}
\]

Identifying \(C/t_k Z\) with \(C^*\) via \(z \mapsto \exp(2\pi iz/t_k)\), we get an action of \(C^*\) on each \(M_{k+1} = \tilde{\Gamma}_{k+1} \setminus R_{k+1}(C)\). The quotient clearly identifies, via \(p_k\), to \(M_k = \Gamma_k \setminus R_k(C)\). Finally, unwinding the definition of \(\{\rho_k\}\) we see that \(\rho_{k-1} = p_k \circ \rho_k\).

(c) Let \(G_{k+1}\) (resp. \(C_{k+1}\)) denote the complex Lie group \(i_k(t_k Z) \setminus R_{k+1}(C)\) (resp. \(i_k(t_k Z) \setminus i_k(C)\)), and let \(\tilde{\Gamma}_{k+1}\) denote the image of \(\Gamma_{k+1}\) under the canonical projection \(R_{k+1}(C) \to G_{k+1}\). Then \(M_{k+1}\) is none other than the quotient of \(G_{k+1}\) by the left action of \(\tilde{\Gamma}_{k+1}\). If \(\Delta_{k+1}\) denotes the group \(\tilde{\Gamma}_{k+1} \cdot C_{k+1}\) in \(G_{k+1}\), then there exists
The space $\mathcal{C}_k$ of holomorphic sections of the line bundle associated to $p_k : M_{k+1} \to M_k$ can now be identified (as a right $G_{k+1}$-module) with the representation of $G_{k+1}$ induced holomorphically by $\mu_{k+1}$. This gives rise to an action $\pi_k$ of $\text{Lie } G_{k+1}$ on $\mathcal{C}_k$.

To define a connection we have to give a way to differentiate the sections in $\mathcal{C}_k$ by the derivations on the base $M_k$. To do this we first note that $\text{Lie } G_{k+1}$ can be realized, when $k > 2$ (resp. $k = 2$), as

$$[y; x_1, x_2, \ldots, x_k] = \begin{pmatrix} 0 & x_1 & x_2 & \cdots & x_k \\ 0 & y & \cdots & y \\ 0 & \cdot & \cdot & \cdot & y \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & 0 \end{pmatrix}_{x_1, \ldots, x_k, y \in \mathbb{C}}$$

(resp. $\{ (y, x) \mid y, x \in \mathbb{C} \}$). We see that the map $\text{Lie } G_{k+1} \to \text{Lie } G_k$ (coming from the differential of $p_k$) admits a vector space- (but not a Lie algebra-) section $s_k$ given, when $k > 2$ (resp. $k = 2$), by $[y; x_1, \ldots, x_{k-1}] \mapsto [y; x_1, \ldots, x_{k-1}, 0]$ (resp. $(y, x) \mapsto [y; x, 0]$). Now define $\nabla_k$ to be the holomorphic connection defined by the action on $\mathcal{C}_k$ via $\pi_k \circ s_k$ of the right invariant derivations on $G_k$.

It is a simple exercise to verify that, locally on $\mathbb{P}_k - \{0, 1, \infty\}$, a section $\varphi$ to the pullback bundle with connection via $\rho_{k-1}$ is flat if and only if

$$\varphi \cdot \ln_{k-1}(x) \, dL(x) + d\varphi = 0.$$

Certainly, $\rho_k$ gives rise to such a (flat) section. Q.E.D.

References


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