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JUMPING TO A UNIFORM UPPER BOUND

HAROLD HODES

ABSTRACT. A uniform upper bound on a class of Turing degrees is the Turing degree of a function which parametrizes the collection of all functions whose degree is in the given class. I prove that if $\underline{a}$ is a uniform upper bound on an ideal of degrees then $\underline{a}$ is the jump of a degree $\underline{c}$ with this additional property: there is a uniform bound $\underline{b} < \underline{a}$ so that $\underline{b} \vee \underline{c} < \underline{a}$.

Fix a recursive pairing function $(x, y) \mapsto \langle x, y \rangle$ from $\omega \times \omega$ onto ω. For $f \in {}^{\omega}\omega$, let $(f)_x(y) = f(\langle x, y \rangle)$; for $\mathcal{F} \subseteq {}^{\omega}\omega$, f parametrizes $\mathcal{F}$ iff $\mathcal{F} = \{(f)_x \mid x \in \omega\}$. Let $\leqslant_T$ and $\equiv_T$ be Turing reducibility and Turing equivalence on ${}^{\omega}\omega$; a degree is an equivalence class under $\equiv_T$. Where I is a set of degrees, a degree $\underline{a}$ is a uniform upper bound (u.u.b.) on I iff some $f \in \underline{a}$ parametrizes $\bigcup I$. I is an ideal iff I is downward closed and closed under join; I is a jump ideal iff it is also closed under jump. Let I be a countable jump ideal and $\underline{a}$ be an upper bound on I. What can we say about degrees which jump to $\underline{a}$? Since $\underline{o}' \in I$, Friedberg's theorem [**2**, p. 265] provides such a $\underline{b}$; but $\underline{b}$ is peculiar in that $\underline{b}' = \underline{o}' \vee \underline{b}$. Since $\underline{o}^{(2)} \in I$, we can relativize to $\underline{o}'$ and obtain $\underline{b} \geqslant \underline{o}'$; but now $b' = \underline{o}^{(2)} \vee \underline{b}$. We would like to have $\underline{c} \vee \underline{b} < \underline{b}' = \underline{a}$ for all $\underline{c} \in I$. In this note we show that if $\underline{a}$ is a u.u.b. on I, we can do this and more.

THEOREM. *If $\underline{a}$ is a u.u.b. on I there are $\underline{b}$ and $\underline{c}$, $\underline{c}$ a u.u.b. on I and $\underline{c} \vee \underline{b} < \underline{b}' = \underline{a}$.*

We prove this by using the trick of [**3**] within a construction like that used in [**1**, Theorem 3]. Fix $f \in \underline{a}$, $A \in \underline{a}$, f parametrizing $\bigcup I$ and $A \in {}^{\omega}2$. Where K is a sequence $\langle n_0, \ldots, n_{l-1} \rangle$; $\hat{K} = \langle (f)_{n_0}, \ldots, (f)_{n_{l-1}} \rangle$. If $u = \langle g_0, \ldots, g_{l-1} \rangle$ is a sequence of functions in ${}^{\omega}\omega$, $\hat{u}$ is the partial function given by $\hat{u}(\langle i, x \rangle) = g_i(x)$ for $i < l$; $u^{\vee}$ is the total extension of $\hat{u}$ such that $u^{\vee}(\langle i, x \rangle) = 0$ for $i \geqslant 0$. We force with the language of arithmetic supplemented by the uninterpreted function symbol '$\underline{g}$' and predicate '$\underline{B}$'. A condition is a pair $\langle K, T \rangle$, where $K \in \omega^{<\omega}$ and T is a total recursive perfect tree represented by its Gödel number. $\langle K, T \rangle \Vdash g(\underline{x}) = \underline{y}$ iff $(x)_0 < \mathrm{lh}(K)$ and $(f)_{(x)_0}((x)_1) = y$; $\langle K, T \rangle \Vdash \underline{B}(\underline{x})$ iff $x < \mathrm{lh}T(\langle\ \rangle)$ and $T(\langle\ \rangle)(x) = 1$. A sequence δ is compatible with K iff $\hat{\hat{K}} \cup \delta$ is a function (viewing δ as a function on $\mathrm{lh}(\delta)$); K floods δ iff $\delta \subseteq \hat{\hat{K}}$. Note that if δ is compatible with K, some extension of K floods δ. Recall Sasso's starred subtrees: $T^*(\delta) = T(\delta^*)$ where $\delta^* = \langle (\delta)_0, 0, (\delta)_1, \ldots, (\delta)_{\mathrm{lh}(\delta)-1}, 0 \rangle$ for $\delta \in \mathrm{Str}$.

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Here is how we would like our construction to proceed. $K_0 = \langle\ \rangle$, $T_0 = \mathrm{id} \restriction \mathrm{Str}$. Suppose we have $\langle K_{2j}, T_{2j} \rangle$. Stage $2j+1$: Fix $z = z(T_{2j})$, depending uniformly on (the Gödel number of) T_{2j} such that for all $B \in [T_{2j}]$, $\{z\}^B(z)\downarrow$ iff $B \notin [T^*_{2j}]$.

Case 1. There are $\delta \in \mathrm{Str}$ and ρ a finite sequence compatible with K_{2j} so that $\{s\}^{T_{2j}(\delta^*)\oplus(\hat{K}_{2j}\cup\rho)}(z) = 0$. Let $\langle \delta_{2j}, \rho \rangle$ be the least such. Let K_{2j+1} be the least extension of K_{2j} flooding ρ. Let $T_{2j+1}(\tau) = T_{2j}(\delta^{*\frown}_{2j} \langle 1,1 \rangle^\frown \tau)$.

Case 2. Otherwise. Let $K_{2j+1} = K_{2j}$ and $T_{2j+1} = T^*_{2j}$.

Stage $2j+2$. $K_{2j+2} = K_{2j+1}{}^\frown \langle j \rangle$. *Case* 1. There is a $\delta \in \mathrm{Str}$ so that $\{j\}^{T_{2j+1}(\delta)}(j)\downarrow$. Let δ_{2j+2} be the least such δ. *Case* 2. Otherwise; let $\delta_{2j+2} = \langle\ \rangle$. Let $T_{2j+2}(\tau) = T_{2j+1}(\delta_{2j+2}{}^\frown \langle A(j) \rangle^\frown \tau)$ for all $\tau \in \mathrm{Str}$.

Let $g = \bigcup_i \hat{K}_i$; $B = \bigcap_i [T_i]$. Clearly g parametrizes $\bigcup I$. We selected T_{2j+1} to meet the requirement $B' \neq \{j\}^{B\oplus g}$. For if Case 1 obtained at stage $2j+1$, $B \notin [T^*_{2j}]$; so $\{z\}^B(z)\downarrow$, so $B'(z) = 1$; but we have chosen K_{2j+1} to make sure that $\{j\}^{B\oplus g}(z) = 0$. If Case 2 obtained, $B \in [T^*_{2j}]$; so $\{z\}^B(z)\uparrow$; so $B'(z) = 0$; but either $\{j\}^{B\oplus g}(z)\uparrow$ or it converges to something different from 0. To compute A from B' we must recover $\langle T_s \rangle_{s\in\omega}$ recursively in B', as a sequence of Gödel numbers. Suppose we have T_{2j}. We have arranged to have B signal to B' the choice of case and the value of δ_{2j+1} in Case 1. For B' can tell whether $B \in [T^*_{2j}]$. If not, we find the longest δ such that $T_{2j}(\delta^*)$ is an initial segment of B; this is δ_{2j+1}; we can now obtain T_{2j+1}. If $B \in [T^*_{2j}]$, we know that $T_{2j+1} = T_{2j}$. An oracle for B and $0'$, which B' provides, suffices for carrying out even steps, that is, obtaining δ_{2j+2}; from this we get $A(j) = i$ iff $T_{2j+1}(\delta_{2j+2}{}^\frown \langle i \rangle)$ is compatible with B; so we recover $A(j)$, and thus get T_{2j+2}. Furthermore B' is recursive in the entire construction. The only hitch is that the construction just described is not recursive in f. For at stage $2j+1$ we needed to answer a Σ^0_1 question about K_{2j}.

Fortunately, since f is a parametrization of $\bigcup I$ (and not just $\bigcup I \cap {}^\omega 2$), we can guess at an m such that $(f)_m = (\hat{K}^\vee_{2j})'$, so that from a certain point on our guesses settle and are right. So we modify the previous construction using the guessing technique of [**1**]. Working on requirement $B' \neq \{j\}^{B\oplus g}$, we may guess that we are in Case 2, when actually we are in Case 1; so we may end up with $B'(z) = 1$ and, contrary to our intentions, $\{j\}^{B\oplus g}(z) = 1$, where $z = z(T)$ and T is the tree which we thought would meet the requirement. But then we shall just go back and attack that requirement again. At the end of stage s we shall have $K^s_0, \ldots, K^s_{d(s)}$, our guesses at $K_0, \ldots, K_{d(s)}$, a tree T_s (no guessing here!), a sequence ρ_s: $s \to \omega$ which is the portion of g to which we are definitely committed, and a number $h(s)$ which tells us how far we have searched for witnesses to being in Case 1 on odd conditions. For $2j+1 < d(s)$, K^s_{2j+1} was instituted by an attack on the requirement $B' = \{j\}^{B\oplus g}$; the stage at which this attack occurred was $t(j,s) \leq s$. Let $c(j,T,K,\tau,q) = 1$ iff our qth guess at $(\hat{K}^\vee)'$ says that there are $\delta \in \mathrm{Str}$ and ρ compatible with K and with τ such that $\{j\}^{T(\delta^*)\oplus(\hat{K}\cup\rho)}(z(T)) = 0$. $c(j,T,K,\tau,q) = 2$ otherwise. (Here τ is a finite sequence of numbers.)

Let $K^0_0 = \langle\ \rangle$, $\rho_0 = \varnothing$, $h(0) = 0$, $d(0) = 0$, $T_0 = \mathrm{id}\,\mathrm{Str}$. We describe stage $s+1$. Suppose we have $K^s_0, \ldots, K^s_{d(s)}$, ρ_s, $h(s)$, T_s, $d(s) = 2j_0$. $2j+1 < d(s)$ is bad at

(s, r) iff $c(j, T_{t(j,s)-1}, K^s_{2j}, \rho_{t(j,s)-1}, h(t(j, s))) = 2$ and $c(j, T_{t(j,s)-1}, K^s_{2j}, \rho_{t(j,s)-1}, h(s) + r + 1) = 1$. In other words, when we instituted K^s_{2j} at stage $t(j, s)$ we thought we were in Case 2, but our $h(s) + r + 1$st guess at $(\hat{K}^{s\vee}_{wj})'$ says that we were in Case 1. Search for the least r such that for some $j \leqslant j_0$:

all $2j' + 1 < 2j + 1$ are nonbad at (s, r);

if $j < j_0$, $2j + 1$ is bad at (s, r);

if no $2j + 1 < d(s)$ is bad at (s, r), $j = j_0$;

if $c(j, T_s, K^s_{2j}, \rho_s, h(s) + r + 1) = 1$, then for some δ and ρ which witness this fact, $\langle \delta, \rho \rangle \leqslant h(s) + r$.

There is such an r, and the least one determines a unique such j. Let $h(s + 1) = h(s) + r + 1$, $d(s + 1) = 2j + 2$, $K^{s+1}_i = K^s_i$ for all $i \leqslant 2j$. We now attack $B' \neq \{j\}^{B\oplus g}$.

Case 1. $c(j, T_s, K^{s+1}_{2j}, \rho_s, h(s + 1)) = 1$. Fix the least $\langle \delta, \rho \rangle$ so that δ and ρ witness this fact. Let K^{s+1}_{2j+1} be the least extension of K^{s+1}_{2j} flooding ρ and ρ_s; let $K^{s+1}_{2j+2} = K^{s+1}_{2j+1}{}^\frown \langle j \rangle$. Let $T^-_{s+1}(\tau) = T_s(\delta^*_{s+1}{}^\frown \langle 1, 1 \rangle^\frown \tau)$ for all $\tau \in \mathrm{Str}$, where $\delta = \delta_{s+1}$.

Case 2. $c(j, T_s, K^{s+1}_{2j}, \rho_s, h(s + 1)) = 2$. Let $K^{s+1}_{2j+1} = K^{s+1}_{2j}$, $K^{s+1}_{2j+2} = K^{s+1}_{2j+1}{}^\frown \langle j \rangle$; let $T^-_{s+1} = T^*_s$. In either case, let ρ_{s+1} be the least extension of ρ_s compatible with K^{s+1}_{2j+2}. If there is a $\delta' \in \mathrm{Str}$ such that $\{s\}^{T^-_{s+1}(\delta')}(s)\downarrow$, let δ'_{s+1} be the least such; otherwise $\delta'_{s+1} = \langle\ \rangle$. $T_{s+1}(\tau) = T^-_{s+1}(\delta'_{s+1}{}^\frown \langle A(s) \rangle^\frown \tau)$ for all $\tau \in \mathrm{Str}$.

LEMMA. *For any j there is a stage $s(j)$ such that for all $s \geqslant s(j)$ and all $j' \leqslant 2j$: $K^s_{j'} = K^{s(j)}_{j'}$, and no requirement $B' \neq \{j'\}^{B\oplus g}$ for $j' < j$ is attacked at stage s.*

Proof of this lemma is routine. Let $K_j = K^{s(j)}_j$, $g = \bigcup_j \hat{K}_j$. Note that $g = \lim_s \rho_s$. If $s + 1$ is the last stage at which $B' \neq \{j\}^{B\oplus g}$ is attacked, for no later s' is j bad at $(s', 0)$; so the requirement is met. As before, $\langle T_s \rangle_{s\in\omega}$ is recursive uniformly in B', and B' can compute A, with δ_{s+1} and δ_{s+1} replacing the δ_{2j+1} and δ'_{2j+2} of our previous attempt. The entire construction is recursive in f, since construction of T_{s+1} from T^-_{s+1} requires only an oracle for $0'$, which f provides, so $B' \leqslant_T f$. Q.E.D.

In general, for what sorts of upper bounds $\underset{\sim}{a}$ on I are there $\underset{\sim}{b}$ so that for all $\underset{\sim}{c} \in I$, $\underset{\sim}{b} \vee \underset{\sim}{c} < \underset{\sim}{b}' = \underset{\sim}{a}$? More pressing, however, is the problem: is every u.u.b. on I (or, for that matter, on the ideal of arithmetic degrees) the jump of an u.b. on I?

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DEPARTMENT OF PHILOSOPHY, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853

Current address: Department of Philosophy, Tufts University, Medford, Massachusetts 02155