BAIRE SECTIONS FOR GROUP HOMOMORPHISMS

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Abstract. The following result is proved: Let $X$ and $Y$ be compact topological groups and $p$ be a continuous group homomorphism from $Y$ onto $X$. Then there exists a map $q$ from $X$ to $Y$ such that $p \circ q = \text{id}_X$ and $q^{-1}(B)$ is a Baire set in $Y$ for every Baire subset $B$ of $X$.

1. Introduction. As pointed out by Rieffel [7], Baire measurable sections for group homomorphisms can be used to construct certain well-behaved extension groups. This motivated Kupka [4] to ask the following question: Given a locally compact group $Y$, a closed subgroup $H$ of $Y$ and the canonical map $p$ from $Y$ onto the space $Y/H$ of left cosets of $Y$, does there exist a Baire measurable map $\varphi: Y/H \to Y$ with $p \circ \varphi = \text{id}_{Y/H}$? We will show that the answer is "yes" provided $Y$ is compact and $H$ is a normal subgroup.

2. Preliminaries. Let $X$ and $Y$ be compact Hausdorff spaces, $\mathcal{B}_0(X)$ and $\mathcal{B}_0(Y)$ their respective Baire $\sigma$-fields. A map $f: X \to Y$ is called Baire measurable iff $f^{-1}(B) \in \mathcal{B}_0(X)$ for all $B \in \mathcal{B}_0(Y)$. A map $\Phi$ from $X$ to the nonempty subsets of $Y$ is said to be a correspondence from $X$ to $Y$ (correspondences are also called multifunctions or set-valued functions in the literature). By $G(\Phi)$ we denote the graph $\{(x, y) \in X \times Y | y \in \Phi(x)\}$ of $\Phi$. $\Phi$ is called upper semi-continuous (u.s.c.) iff, for every open subset $U$ of $Y$, the set $\{x \in X | \Phi(x) \subset U\}$ is open in $X$. A compact-valued correspondence $\Phi$ is u.s.c. if and only if $G(\Phi)$ is closed in $X \times Y$.

A map $f: X \to Y$ is called a selection for $\Phi$ iff $f(x) \in \Phi(x)$ for all $x \in X$. A compact Hausdorff space $X$ is said to have the Bockstein separation property (BSP) iff any two disjoint open subsets of $X$ can be separated by open $\mathcal{G}_\sigma$-sets (cf. Pelczyński [6, Definition 5.9]). A classical theorem of Bockstein [1] states that an arbitrary product of compact metrizable spaces has the BSP. The same is true for compact topological groups (cf. Pelczyński [6, Theorem 7.5 and Corollary 5.11]).

3. A selection lemma. The following lemma will be used in the proof of our main theorem but may also be of some interest in itself.

Lemma. Let $X$ be a compact Hausdorff space with the BSP, $Z$ a compact metrizable space, and $\Phi$ an u.s.c. compact-valued correspondence from $X$ to $Z$. Then $\Phi$ has a Baire measurable selection.
Proof. We first note that, due to the fact that $X$ has the BSP, the following holds.

\[(*)\] For every subset $F$ of $X$ the set $\hat{F}$ is a Baire set

(where $\overline{A}$ and $\dot{A}$ denote the closure and the interior of a set $A$ respectively). To show this let $F$ be a subset of $X$. BSP implies that there is an open Baire set $B$ such that $\hat{F} \subset B$ and $B \cap (X \setminus \hat{F}) = \emptyset$. This implies $\hat{F} \subset B \subset \overline{F}$, hence $\hat{F} = B$ because $B$ is open.

We will now show that there is a compact-valued correspondence $\Phi$ from $X$ to $Z$ such that

(i) $\phi(x)$ is a subset of $\Phi(x)$ for all $x \in X$,

(ii) $\{x \in X \mid \phi(x) \cap A \neq \emptyset\}$ is a Baire subset of $X$ for all closed subsets $A$ of $Z$.

Suppose for the moment that there is such a $\phi$. Then the selection theorem of Kuratowski and Ryll-Nardzewski [5] implies that $\phi$ has a Baire measurable selection. Since such a selection is also a selection for $\Phi$ our lemma will follow.

To construct $\phi$ we define for each $x$ in $X$ a collection $\mathcal{F}_x$ of nonempty subsets of $Z$ by

$$\mathcal{F}_x := \{ B \subset Z \mid B \text{ open, } x \in \Phi^{-1}(B) \}$$

where $\Phi^{-1}(B) := \{ x \in X \mid \Phi(x) \subset B \}$. We claim that $\mathcal{F}_x$ has the finite intersection property. This is a consequence of the following facts:

1. $\mathcal{F}_x(F \cap G) = \mathcal{F}_x(F) \cap \mathcal{F}_x(G)$ for any two subsets $F$ and $G$ of $Z$,

2. $\mathcal{F}_x(U_1 \cap U_2) = \mathcal{F}_x(U_1) \cap \mathcal{F}_x(U_2)$ for any two open subsets $U_1$ and $U_2$ of $X$,

3. $\mathcal{F}_x(U)$ is open for every open subset $U$ of $Z$ because $\Phi$ is u.s.c.

Therefore $\Phi(x) := \bigcap \{ B \mid B \in \mathcal{F}_x \}$ defines a compact-valued correspondence $\Phi$ from $X$ to $Z$.

To show that $\Phi$ satisfies (i), assume that there are $x$ in $X$ and $z$ in $Z$ such that $z \in \Phi(x) \setminus \Phi(x)$. Because $Z$ is regular there is an open neighborhood $U$ of $z$ with $U \cap \Phi(x) = \emptyset$. This implies $x \in \Phi^{-1}(Z \setminus U)$, hence $Z \setminus U \in \mathcal{F}_x$ and therefore $z \in U \cap (Z \setminus U) \subset U \cap (Z \setminus U) = U \cap (Z \setminus U) = \emptyset$ which is absurd.

(ii) is equivalent to

(ii') $\Phi^{-1}(U)$ is a Baire set for every open subset $U$ of $Z$.

So let $U \subset Z$ be open. Since $Z$ is metrizable there exists an increasing sequence $(B_n)_{n \in \mathbb{N}}$ of open sets such that $\bigcup_n B_n = \bigcup_n \overline{B_n} = U$. We show that

$$\Phi^{-1}(U) = \bigcup_n \overline{\Phi^{-1}(B_n)}$$

holds, from which (ii') will follow because each of the sets $\overline{\Phi^{-1}(B_n)}$ is a Baire set by ($\ast$).

For $x \in \Phi^{-1}(B_n)$ we have $B_n \in \mathcal{F}_x$, hence $\Phi(x) \subset \overline{B_n} \subset U$, which proves one of the required inclusions. To prove the other one let $\Phi(x)$ be contained in $U$. This
implies that $B \subset U$ holds for some $B \in \mathcal{F}_X$. $B$ being compact there is an $n \in \mathbb{N}$ with $B \subset B_n$. Therefore $x \in \Phi_{-1}(B) \subset \Phi_{-1}(B_n)$ and the selection lemma is proved.

**Remarks.** (1) Note that in the situation of the lemma the inverse image $\{x \in X | \Phi(x) \cap A \neq \emptyset\}$ of a closed set $A \subset Z$ under $\Phi$ need not be Baire measurable. Therefore, the theorem of Kuratowski and Ryll-Nardzewski applied to $\Phi$, in general only yields a Borel measurable selection for $\Phi$.

(2) The lemma, even in a slightly more general form, can also be derived from the main theorem in [2, Theorem 1, p. 343]. The proof given here uses methods similar to those employed in proving that general theorem.

4. Main results. In this section we will establish a selection theorem for correspondences whose graphs are groups. The main ingredients of the proof are the selection lemma and the fact that compact groups have the BSP.

**Theorem.** Let $X$ and $Y$ be compact topological groups and $\Phi$ an u.s.c. compact-valued correspondence from $X$ to $Y$ such that $G(\Phi)$ is a subgroup of the product group $X \times Y$. Then $\Phi$ has a Baire measurable selection.

**Proof.** (a) First we consider the case $Y = \prod_{j \in J} Y_j$, where each $Y_j$ is a compact metrizable group. For $J \subset I$ let $Y_J = \prod_{j \in J} Y_j$ and $\pi_j: Y \to Y_j$, $\hat{\pi}_j: X \times Y \to X \times Y_J$ be the canonical projections. Let $\Phi_j$ be the correspondence from $X$ to $Y_j$ defined by $\Phi_j(x) = \pi_j(\Phi(x))$. Then we have $G(\Phi_j) = \hat{\pi}_j(G(\Phi))$, hence $G(\Phi_j)$ is a compact subgroup of $X \times Y_J$ because $\hat{\pi}_j$ is a continuous group homomorphism. In particular, $G(\Phi_j)$ has the BSP. Now let $\Gamma = \{(J, \varphi) | J \subset I, J \neq \emptyset, \varphi: X \to Y_J \text{ Baire measurable selection of } \Phi_J\}$.

We introduce a partial order $\preceq$ on $\Gamma$ by

$$(J, \varphi) \preceq (K, \psi) \text{ iff } J \subset K \text{ and } \pi_j \circ \varphi = \pi_j \circ \psi \text{ for all } j \in J$$

and claim that $\Gamma$ is nonempty and inductively ordered by $\preceq$. For $i \in I$ the correspondence $\Phi_i$ is u.s.c. and takes compact values in the compact metrizable space $Y_i$. Hence, by the selection lemma, $\Phi_i$ admits a Baire measurable selection $\varphi_i$, i.e. $((i), \varphi_i) \in \Gamma$. Now let $(J_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ be a chain in $\Gamma$. Let $J = \bigcup J_\lambda$ and define $\varphi: X \to Y_J$ by $\pi_j \varphi(x) = \pi_j \varphi_\lambda(x)$, if $j \in J_\lambda$.

Then $\varphi$ is a well-defined map. The definition of $\varphi$ and the Baire measurability of the $\varphi_\lambda$'s implies that for each $j \in J$ the map $\pi_j \circ \varphi$ is Baire measurable. Since the Baire $\sigma$-algebra on $Y_J$ is the smallest $\sigma$-algebra rendering all the maps $\pi_j$ measurable, it follows that $\varphi$ is Baire measurable. Therefore $(J, \varphi)$ is an upper bound of $(J_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ in $\Gamma$. By Zorn's lemma there exists a maximal element $(M, \mu)$ in $\Gamma$. To complete the proof of (a) it remains to show $M = I$. Assume the contrary. Then there is a $j \in I \setminus M$. Define a correspondence $\Psi$ from $G(\Phi)$ to $Y_J$ by

$$\Psi((x, y)) = \{z \in Y_J | (y, z) \in \Phi_{M \cup \{j\}}(x)\}.$$ 

The graph of $\Psi$ is equal to $G(\Phi_{M \cup \{j\}})$, hence compact. This implies that $\Psi$ is u.s.c. and compact-valued. Since $G(\Phi_M)$ has the BSP, the selection lemma yields a Baire measurable selection $\psi$ for $\Psi$. Define $\varphi: X \to Y_{M \cup \{j\}}$ by $\varphi(x) = (\mu(x), \psi(x, \mu(x)))$. 


Then \( \varphi \) is obviously a selection for \( \Phi_{M \cup \{j\}} \). To show that \( \varphi \) is Baire measurable we have to check the measurability of the maps \( \pi_i \circ \varphi \) with \( i \in M \cup \{j\} \). For \( i \in M \) it follows from \( \pi_i \circ \varphi = \pi_i \circ \mu \). Moreover, we have \( \pi_j \varphi(x) = \psi(x, \mu(x)) \) for all \( x \in X \).
Since \( x \mapsto (x, \mu(x)) \) is Baire measurable as a map into \( X \times Y_M \) taking values in \( G(\Phi_M) \), it is also Baire measurable as a map into \( G(\Phi_M) \) because \( G(\Phi_M) \) is compact. Hence \( \pi_j \circ \varphi \) is Baire measurable as a composition of Baire measurable maps. Thus \( (M \cup \{j\}, \varphi) \) is an element of \( \Gamma \) strictly larger than the maximal element \( (M, \mu) \), a contradiction.

(b) To prove the general case we observe that every compact topological group \( Y \) is a subgroup of a product \( \Pi Y_i \) of compact metrizable groups \( Y_i \), because it is a projective limit of such groups (cf. e.g. Higgins [3, p. 98, Theorem A'']). Hence by (a) there exists a selection \( \varphi \) of \( \Phi \) which is Baire measurable as a map into \( \Pi Y_i \). As before we see that it is also Baire measurable as a map into \( Y \). Hence the theorem follows.

Important examples of correspondences satisfying the assumptions of our theorem are given by \( \Phi = p'x \) where \( p \) is a continuous homomorphism from one compact group onto another. This immediately leads to the following corollary.

**Corollary.** Let \( X \) and \( Y \) be compact topological groups and \( p: Y \to X \) a continuous surjective homomorphism. Then there exists a Baire measurable map \( \varphi: X \to Y \) with \( p \circ \varphi = \text{id}_X \).

In particular the result announced in the introduction holds.

**Remark.** The map \( \varphi \) in the corollary can be chosen in such a way that it maps the identity element onto the identity element (define a new section by \( x \mapsto \varphi(e)^{-1}\varphi(x) \)). Therefore one always has measurable cross sections in the sense of Rieffel [7, p. 872], provided the groups involved are compact.

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**Added in proof.** Using the same methods, it can be shown that the answer to Kupka's question—mentioned in the introduction—remains "yes" even if the normality condition on the subgroup \( H \) is dropped.

**References**


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