BAIRE SECTIONS FOR GROUP HOMOMORPHISMS

S. GRAF AND G. MÄGERL

Abstract. The following result is proved: Let X and Y be compact topological groups and p be a continuous group homomorphism from Y onto X. Then there exists a map q from X to Y such that p \circ q = \text{id}_X and q^{-1}(B) is a Baire set in Y for every Baire subset B of X.

1. Introduction. As pointed out by Rieffel [7], Baire measurable sections for group homomorphisms can be used to construct certain well-behaved extension groups. This motivated Kupka [4] to ask the following question: Given a locally compact group Y, a closed subgroup H of Y and the canonical map p from Y onto the space Y/H of left cosets of Y, does there exist a Baire measurable map \( \varphi : Y/H \rightarrow Y \) with \( p \circ \varphi = \text{id}_{Y/H} \)? We will show that the answer is "yes" provided Y is compact and H is a normal subgroup.

2. Preliminaries. Let X and Y be compact Hausdorff spaces, \( \mathcal{B}_0(X) \) and \( \mathcal{B}_0(Y) \) their respective Baire \( \sigma \)-fields. A map \( f : X \rightarrow Y \) is called Baire measurable iff \( f^{-1}(B) \in \mathcal{B}_0(X) \) for all \( B \in \mathcal{B}_0(Y) \). A map \( \Phi \) from X to the nonempty subsets of Y is said to be a correspondence from X to Y (correspondences are also called multifunctions or set-valued functions in the literature). By \( G(\Phi) \) we denote the graph \( \{(x, y) \in X \times Y \mid y \in \Phi(x)\} \) of \( \Phi \). \( \Phi \) is called upper semi-continuous (u.s.c.) iff, for every open subset \( U \) of Y, the set \( \{x \in X \mid \Phi(x) \subset U\} \) is open in X. A compact-valued correspondence \( \Phi \) is u.s.c. if and only if \( G(\Phi) \) is closed in \( X \times Y \).

A map \( f : X \rightarrow Y \) is called a selection for \( \Phi \) iff \( f(x) \in \Phi(x) \) for all \( x \in X \). A compact Hausdorff space X is said to have the Bockstein separation property (BSP) iff any two disjoint open subsets of X can be separated by open \( \mathcal{F}_\sigma \)-sets (cf. Pełczyński [6, Definition 5.9]). A classical theorem of Bockstein [1] states that an arbitrary product of compact metrizable spaces has the BSP. The same is true for compact topological groups (cf. Pełczyński [6, Theorem 7.5 and Corollary 5.11]).

3. A selection lemma. The following lemma will be used in the proof of our main theorem but may also be of some interest in itself.

Lemma. Let X be a compact Hausdorff space with the BSP, Z a compact metrizable space, and \( \Phi \) an u.s.c. compact-valued correspondence from X to Z. Then \( \Phi \) has a Baire measurable selection.
PROOF. We first note that, due to the fact that $X$ has the BSP, the following holds.

\((\ast)\)

For every subset $F$ of $X$ the set $\hat{F}$ is a Baire set

(where $\overline{A}$ and $\mathring{A}$ denote the closure and the interior of a set $A$ respectively). To show this let $F$ be a subset of $X$. BSP implies that there is an open Baire set $B$ such that $\hat{F} \subset B$ and $B \cap (X \setminus \hat{F}) = \emptyset$. This implies $\hat{F} \subset B \subset \hat{F}$, hence $\hat{F} = B$ because $B$ is open.

We will now show that there is a compact-valued correspondence $\Phi$ from $X$ to $Z$ such that

(i) $\Phi(x)$ is a subset of $\Phi(x)$ for all $x$ in $X$,

(ii) $\{x \in X \mid \Phi(x) \cap A \neq \emptyset\}$ is a Baire subset of $X$ for all closed subsets $A$ of $Z$.

Suppose for the moment that there is such a $\Phi$. Then the selection theorem of Kuratowski and Ryll-Nardzewski [5] implies that $\Phi$ has a Baire measurable selection. Since such a selection is also a selection for $\Phi$ our lemma will follow.

To construct $\Phi$ we define for each $x$ in $X$ a collection $\mathcal{F}_x$ of nonempty subsets of $Z$ by

$$\mathcal{F}_x := \left\{ B \subset Z \mid B \text{ open, } x \in \Phi_\ast(B) \right\}$$

where $\Phi_\ast(B) := \{ x \in X \mid \Phi(x) \subset B \}$. We claim that $\mathcal{F}_x$ has the finite intersection property. This is a consequence of the following facts:

1. $\mathcal{F}_x(F \cap G) = \mathcal{F}_x(F) \cap \mathcal{F}_x(G)$ for any two subsets $F$ and $G$ of $Z$,

2. $\hat{U}_1 \cap \hat{U}_2 = \hat{U}_1 \cap \hat{U}_2$ for any two open subsets $U_1$ and $U_2$ of $X$,

3. $\mathcal{F}_x(£)$ is open for every open subset $B$ of $Z$ because $\Phi$ is u.s.c.

Therefore $\Phi(x) := \bigcap \{ \overline{B} \mid B \in \mathcal{F}_x \}$ defines a compact-valued correspondence $\Phi$ from $X$ to $Z$.

To show that $\Phi$ satisfies (i), assume that there are $x$ in $X$ and $z$ in $Z$ such that $z \in \hat{\Phi}(x) \setminus \Phi(x)$. Because $Z$ is regular there is an open neighborhood $U$ of $z$ with $\hat{U} \setminus \Phi(x) = \emptyset$. This implies $x \in \Phi_\ast(Z \setminus \hat{U})$, hence $Z \setminus \hat{U} \subset \mathcal{F}_x$, and therefore $z \in U \cap (Z \setminus U) \subset U \cap (Z \setminus U) = \emptyset$ which is absurd.

(ii) is equivalent to

(ii)' $\Phi_\ast(U)$ is open for every open subset $U$ of $Z$.

So let $U \subset Z$ be open. Since $Z$ is metrizable there exists an increasing sequence $(B_n)_{n \in \mathbb{N}}$ of open sets such that $\bigcup_n B_n = \bigcup_n \overline{B}_n = U$. We show that

$$\Phi_\ast(U) = \bigcup_n \Phi_\ast(B_n)$$

holds, from which (ii)' will follow because each of the sets $\Phi_\ast(B_n)$ is a Baire set by (\ast).

For $x \in \Phi_\ast(B_n)$ we have $B_n \subset \mathcal{F}_x$, hence $\Phi(x) \subset \overline{B}_n \subset U$, which proves one of the required inclusions. To prove the other one let $\Phi(x)$ be contained in $U$. This
implies that \( \overline{B} \subset U \) holds for some \( B \in \mathcal{F}_x \). \( \overline{B} \) being compact there is an \( n \in \mathbb{N} \) with \( \overline{B} \subset B_n \). Therefore \( x \in \Phi^{-1}_x(B) \subset \Phi^{-1}_x(B_n) \) and the selection lemma is proved.

**Remarks.** (1) Note that in the situation of the lemma the inverse image \( \{x \in X | \Phi(x) \cap A \neq \emptyset \} \) of a closed set \( A \subset Z \) under \( \Phi \) need not be Baire measurable. Therefore, the theorem of Kuratowski and Ryll-Nardzewski applied to \( \Phi \), in general only yields a Borel measurable selection for \( \Phi \).

(2) The lemma, even in a slightly more general form, can also be derived from the main theorem in [2, Theorem 1, p. 343]. The proof given here uses methods similar to those employed in proving that general theorem.

4. Main results. In this section we will establish a selection theorem for correspondences whose graphs are groups. The main ingredients of the proof are the selection lemma and the fact that compact groups have the BSP.

**Theorem.** Let \( X \) and \( Y \) be compact topological groups and \( \Phi \) an u.s.c. compact-valued correspondence from \( X \) to \( Y \) such that \( G(\Phi) \) is a subgroup of the product group \( X \times Y \). Then \( \Phi \) has a Baire measurable selection.

**Proof.** (a) First we consider the case \( Y = \prod_{j \in J} Y_j \), where each \( Y_j \) is a compact metrizable group. For \( J \subset I \) let \( Y_J = \prod_{j \in J} Y_j \) and \( \pi_j: Y_J \to Y_j \), \( \hat{\pi}_j: X \times Y_J \to X \times Y_j \) be the canonical projections. Let \( \Phi_j \) be the correspondence from \( X \) to \( Y_j \) defined by \( \Phi_j(x) = \pi_j(\Phi(x)) \). Then we have

\[
G(\Phi_j) = \hat{\pi}_j(G(\Phi)),
\]

hence \( G(\Phi_j) \) is a compact subgroup of \( X \times Y_j \) because \( \hat{\pi}_j \) is a continuous group homomorphism. In particular, \( G(\Phi_j) \) has the BSP. Now let \( \Gamma = \{(J, \varphi) | J \subset I, J \neq \emptyset, \varphi: X \to Y_j \text{ Baire measurable selection of } \Phi_j \} \).

We introduce a partial order \( \preceq \) on \( \Gamma \) by

\[
(J, \varphi) \preceq (K, \psi) \text{ iff } J \subset K \text{ and } \pi_j \circ \varphi = \pi_j \circ \psi \text{ for all } j \in J
\]

and claim that \( \Gamma \) is nonempty and inductively ordered by \( \preceq \). For \( i \in I \) the correspondence \( \Phi_i \) is u.s.c. and takes compact values in the compact metrizable space \( Y_i \). Hence, by the selection lemma, \( \Phi_i \) admits a Baire measurable selection \( \varphi_i \), i.e.

\[
(\{i\}, \varphi_i) \in \Gamma.
\]

Now let \( (J_\lambda, \varphi_\lambda)_{\lambda \in \Lambda} \) be a chain in \( \Gamma \). Let \( J = \bigcup J_\lambda \) and define \( \varphi: X \to Y_J \) by \( \varphi(x) = \pi_j(\Phi(x)) \), if \( j \in J_\lambda \).

Then \( \varphi \) is a well-defined map. The definition of \( \varphi \) and the Baire measurability of the \( \varphi_\lambda \)'s implies that for each \( j \in J \) the map \( \pi_j \circ \varphi \) is Baire measurable. Since the Baire \( \sigma \)-algebra on \( Y_J \) is the smallest \( \sigma \)-algebra rendering all the maps \( \pi_j \) measurable, it follows that \( \varphi \) is Baire measurable. Therefore \( (J, \varphi) \) is an upper bound of \( (J_\lambda, \varphi_\lambda)_{\lambda \in \Lambda} \) in \( \Gamma \). By Zorn's lemma there exists a maximal element \( (M, \mu) \) in \( \Gamma \). To complete the proof of (a) it remains to show \( M = I \). Assume the contrary. Then there is a \( j \in I \setminus M \). Define a correspondence \( \Psi \) from \( G(\Phi) \) to \( Y_J \) by

\[
\Psi((x, y)) = \{z \in Y_J | (y, z) \in \Phi_{M \cup \{j\}}(x)\}.
\]

The graph of \( \Psi \) is equal to \( G(\Phi_{M \cup \{j\}}) \), hence compact. This implies that \( \Psi \) is u.s.c. and compact-valued. Since \( G(\Phi_M) \) has the BSP, the selection lemma yields a Baire measurable selection \( \psi \) for \( \Psi \). Define \( \varphi: X \to Y_{M \cup \{j\}} \) by \( \varphi(x) = (\mu(x), \psi(x, \mu(x))) \).
Then \( \varphi \) is obviously a selection for \( \Phi_{\mu(j)} \). To show that \( \varphi \) is Baire measurable we have to check the measurability of the maps \( \pi_i \circ \varphi \) with \( i \in M \cup \{j\} \). For \( i \in M \) it follows from \( \pi_i \circ \varphi = \pi_i \circ \mu \). Moreover, we have \( \pi_j \varphi(x) = \psi(x, \mu(x)) \) for all \( x \in X \). Since \( x \mapsto (x, \mu(x)) \) is Baire measurable as a map into \( X \times Y_M \) taking values in \( G(\Phi_M) \), it is also Baire measurable as a map into \( G(\Phi_M) \) because \( G(\Phi_M) \) is compact. Hence \( \pi_j \circ \varphi \) is Baire measurable as a composition of Baire measurable maps. Thus \( (M \cup \{j\}, \varphi) \) is an element of \( \Gamma \) strictly larger than the maximal element \( (M, \mu) \), a contradiction.

(b) To prove the general case we observe that every compact topological group \( Y \) is a subgroup of a product \( \Pi Y_j \) of compact metrizable groups \( Y_j \), because it is a projective limit of such groups (cf. e.g. Higgins [3, p. 98, Theorem A'']). Hence by (a) there exists a selection \( \varphi \) of \( \Phi \) which is Baire measurable as a map into \( \Pi Y \). As before we see that it is also Baire measurable as a map into \( Y \). Hence the theorem follows.

Important examples of correspondences satisfying the assumptions of our theorem are given by \( \Phi = \pi'x \) where \( \pi \) is a continuous homomorphism from one compact group onto another. This immediately leads to the following corollary.

**Corollary.** Let \( X \) and \( Y \) be compact topological groups and \( \pi: Y \to X \) a continuous surjective homomorphism. Then there exists a Baire measurable map \( \varphi: X \to Y \) with \( \pi \circ \varphi = \text{id}_X \).

In particular the result announced in the introduction holds.

**Remark.** The map \( \varphi \) in the corollary can be chosen in such a way that it maps the identity element onto the identity element (define a new section by \( x \mapsto \varphi(e)^{-1}\varphi(x) \)). Therefore one always has measurable cross sections in the sense of Rieffel [7, p. 872], provided the groups involved are compact.

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**Added in proof.** Using the same methods, it can be shown that the answer to Kupka's question—mentioned in the introduction—remains "yes" even if the normality condition on the subgroup \( H \) is dropped.

**References**