GRADED ALGEBRAS HAVING
A UNIQUE RATIONAL HOMOTOPY TYPE

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ABSTRACT. We consider the problem for which graded algebra A^* complexes such that $H^*(K, Q) \simeq A^*$ is unique up to homotopy type. A necessary and sufficient condition is given using formal minimal model of A^*.

1. Introduction. Rational homotopy types are homotopy types of localized spaces at zero. In [2, 6] Sullivan constructed the theory of minimal models which algebraically describes rational homotopy types.

It was shown by Quillen, Sullivan and others [2, 4] that there exists a complex K with $H^*(K; Q) \simeq A^*$ for any simply connected commutative graded algebra A^* over Q.

We consider the problem, for which algebra A^*, the complex K such that $H^*(K, Q) \simeq A^*$ is determined uniquely up to rational homotopy type. The uniqueness of the rational homotopy type of A^* was shown by Mimura and Toda [3] when A^* is a tensor product of truncated polynomial algebras and by Body and Sullivan [1, 7] when A^* has a regular set of relations.

In this paper, when A^* is of finite type, we shall show that it can be known, by means of the minimal model of A^* with $d = 0$, whether such rational homotopy type is unique or not.

In §2, we recall the definition and the construction of minimal models. In §3, we introduce condition (C) which roughly says that, for some n, we can choose a ring homomorphism

$$\sigma_{n-1} \colon H^*(m(A)(n-1)) \to A^*,$$

such that the $n + 1$ dimensional kernel of σ_{n-1} cannot be transformed to that of ρ_A^* by any automorphism of $m(A)(n-1)$, and we can follow the way of construction of minimal models based on $m(A)(n-1)$ and σ_{n-1}. For the definition $m(A)(n-1)$ and ρ_A see §2. This condition gives the possibility of the existence of models which are not isomorphic to the formal one and will be shown to be necessary and sufficient.

In §4, some examples are given. For example consider the algebras

$$A_1^* = \bigwedge (a, b, c, x) / (a^2bc, ab^2c, abc^2, I),$$

$$A_2^* = \bigwedge (a, b, c, x) / (a^2bc, ab^2c, abc^2, ax, I)$$

where $|a|$, $|b|$, $|c|$ are even with $|a| < |b| < |c|$, and $|x| = |a| + 2|b| + 2|c|$. Here $|*|$ means the degree of $*$ and I denotes the ideal generated by all elements of degree $> |ax|$. Then we can show by using condition (C) that A_1^* has unique homotopy type and A_2^* has plural types. In §5, algebras which are uniquely determined are studied, and we see uniqueness depends essentially on degree. In §6, some remarks on functorial properties are given.

After writing this paper, we received a letter and a paper [8] from Professor James Stasheff which contains many suggestions; he pointed out insufficiency of the first version of the condition (C). We would like to take this opportunity to thank him.

2. Preliminaries to minimal models. Throughout this paper we consider only G.A. (commutative graded algebra) A^* over Q with $A^0 = Q$, $A^1 = 0$. A free G.A. means a tensor product of a polynomial algebra of even dimensional elements and an exterior algebra of odd dimensional elements, and we denote by $\wedge(x_1, \ldots, x_n)$ the free algebra generated by $(x_1, \ldots, x_n)$. For each D.G.A. (differential G.A.) A^* there is a D.G.A. $m^*(A^*)$, the minimal model of A^*, which satisfies the following three conditions:

(1) $m^*(A^*)$ is free;

(2) $d(m^*(A^*))$ is decomposable;

(3) there is a D.G.A. map $\rho: m^*(A^*) \to A^*$ such that $\rho^*: H^*(m^*(A^*)) \simeq H^*(A^*)$.

Let $m^*(A^*)(n)$ denote the subalgebra of $m(A^*)$ generated by elements of degree $\leq n$. Then $m(A^*)$ is constructed by $m(A^*)(n)$ inductively as follows [2]:

$$m(A^*)(n) = m(A^*)(n-1)\{x_k, \eta_m\},$$

where $\{\rho(x_k)\}$ forms a basis for the cokernel of

$$\rho^n: H^n(m(A^*)(n-1)) \to H^n(A^*)$$

and $\{d\eta_m\}$ forms a basis for the kernel of

$$\rho^{n+1}: H^{n+1}(m(A^*)(n-1)) \to H^{n+1}(A^*)$$

and $\rho(\eta_m) = z_m$, where $dz_m = \rho(d\eta_m)$.

Sullivan [2, 6] showed that rational homotopy types are in 1-1 correspondence with minimal models of D.G.A. via the Q-polynomial differential forms. In this paper we shall study the condition of unique existence of $m(B^*)$ with $H^*(m(B)) \simeq A^*$ for a given G.A. A^*.

3. A necessary and sufficient condition. For G.A. A^*, we can construct the minimal model of A^*, regarded as a D.G.A., with differential $d = 0$. Using this minimal model $m(A^*)$, we shall see when the complex K, with $H^*(K, Q) = A^*$, is determined uniquely up to rational homotopy type.

Let $\rho_A: m(A^*)(k) \to A^*$ be the D.G.A. map constructed in §2. We consider the following condition:

(C_1) For some integer n, there exists a G.A. map

$$\sigma_{k-1}: H^*(m(A)(n-1)) \to A^*$$

such that $\sigma_{n-1} = \rho_A^*$ for $* \leq n-1$, and there is no D.G.A. automorphism

$$\chi: m(A)(n-1) \to m(A)(n-1)$$

satisfying

$$\chi^*(\ker \sigma_{n-1})^{n+1} = (\ker \rho_A^*)^{n+1}.$$

Let m_n be a D.G.A. obtained from $m(A)(n-1)$ by adding the cokernel of

$$\sigma_{n-1}: H^n(m(A)(n-1)) \to A^n$$

and the kernel of

$$\sigma_{n-1}: H^{n+1}(m(A)(n-1)) \to A^{n+1}.$$

Then we show

LEMMA 3.1. *The following are equivalent*:
(a) (C_1) *is satisfied*;
(b) m_n *and* $m(A)(n)$ *are not isomorphic*.

PROOF. (a) $\Rightarrow$ (b). Suppose we have the isomorphism $\varphi: m_n \to m(A)(n)$. Put $\varphi \mid m(A)(n-1) = \chi$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
H^{n+1}(m_n) & \overset{\sim}{\underset{\varphi^*}{\to}} & H^{n+1}(m(A)(n)) \\
\uparrow i^* & & \uparrow j^* \\
H^{n+1}(m(A)(n-1)) & \overset{\sim}{\underset{\chi^*}{\to}} & H^{n+1}(m(A)(n-1))
\end{array}
$$

where i and j are inclusions. Since $(\ker i^*)^{n+1} = (\ker \sigma_{n-1})^{n+1}$ and $(\ker j^*)^{n+1} = (\ker \rho_A^*)^{n+1}$, from the above diagram, we have $\chi^*(\ker \sigma_{n-1})^{n+1} = (\ker \rho_A^*)^{n+1}$.

(b) $\Rightarrow$ (a). Since all elements of $H^n(m(A)(n-1))$ are decomposable, σ_{n-1} and ρ_A^* are identical on $H^n(m(A)(n-1))$. Therefore $(\text{coker } \sigma_{n-1})^n = (\text{coker } \rho_A^*)^n$. If $\chi^*(\ker \sigma_{n-1})^{n+1} = (\ker \rho_A^*)^{n+1}$, we can extend χ to an isomorphism $m_n \to m(A)(n)$. Q.E.D.

Put $m_{n-1} = m(A)(n-1)$. If σ_{k-1} and m_{k-1} are defined, we define m_k inductively by

$$m_k = m_{k-1} \otimes \wedge (y_1, \ldots, y_s, \eta_1, \ldots, \eta_t),$$

where $\{y_i\}$ forms a basis for the cokernel of σ_{k-1} in degree k and $\{d\eta_j\}$ forms a basis for the kernel of σ_{k-1} in degree $k+1$.

$(C_2)_k$ There exists a G.A. map σ_k so that the following diagram commutes:

$$
\begin{array}{ccc}
H^*(m_k) & \xrightarrow{\sigma_k} & \\
\uparrow i_{k-1}^* & & \searrow A^* \\
H^*(m_{k-1}) & \xrightarrow{\sigma_{k-1}} &
\end{array}
$$

where $i_{k-1}: m_{k-1} \to m_k$ is an inclusion and $\sigma_k([y_k]) = y_k$.

Note that σ_k is then isomorphic for $* \leqslant k$ and monomorphic for $* = k+1$. If (C_1) and $(C_2)_k$ are satisfied for each $k \geqslant n$, we say that condition (C) is satisfied.

If $m(A)$ satisfies the condition (C), then $m_\infty = \bigcup_k m_k$ is a free D.G.A. with $H^*(m_\infty) \simeq A^*$ and dm_∞^+ are decomposed. Hence m_∞ is a minimal model of itself.

The minimal models m_∞ and $m(A)$ are not isomorphic because $m_\infty(n) = m_n$ and $m(A)(n)$ are not isomorphic by Lemma 3.1. This shows that A^* has plural homotopy types.

Conversely if we have a D.G.A. $\bar{m}$ with $H^*(\bar{m}) \cong A^*$ which is not isomorphic to $m(A)$, then there is an integer n such that $\bar{m}(n-1)$ and $m(A)(n-1)$ are isomorphic but $\bar{m}(n)$ and $m(A)(n)$ are not. Then we define σ_k by the composition

$$H^*(\bar{m}(k)) \underset{i^*}{\to} H^*(\bar{m}) \overset{\sim}{\to} A^*$$

where i is the inclusion. By Lemma 3.1, condition (C_1) is satisfied for σ_{n-1} and $(C_2)_k$ is also satisfied for each $k \geq n$ from the definition of σ_k. Thus we obtain

THEOREM 3.2. *A^* has plural homotopy types if and only if the condition* (C) *is satisfied.*

Now we rewrite the condition (C) in terms of generators of $m(A)$. We fix generators of A^* and $m(A)$ as follows:

(1) $A^* = \wedge(x_1, \ldots, x_s)/(I)$ where I is an ideal;

(2) $m(A) = \wedge(x_1, \ldots, x_s, \theta_1, \ldots, \theta_t, \xi_1, \xi_2, \ldots)$

where $d\theta_i \in \wedge(x_1, \ldots, x_s)$, $d\xi_j \notin \wedge(x_1, \ldots, x_s)$. Here we use the same notation x_i for closed generators of $m(A)$ as for those of A^*.

Let $\{[d\xi_i], [d\theta_j]\}_{i \in I, j \in J}$ be a basis of $(\ker \rho_A^*)^{n+1}$, and let

$$\sigma_{n-1} \colon H^*(m(A)(n-1)) \to A^*$$

be a G.A. map such that $\sigma_{n-1} = \rho_A^*$ for $* \leq n-1$. Set $I' = \{i \in I \mid \sigma_{n-1}(d\xi_i)$ is decomposable in $A^*\}$.

LEMMA 3.3. $\{[d\xi_i - P_i], [d\theta_j]\}_{i \in I', j \in J}$ *forms a basis for the* $(\ker \sigma_{n-1})^{n+1}$, *where* P_i *is any element in* $\wedge(x_1, \ldots, x_s)$ *such that* $\rho_A(P_i) = \sigma_{n-1}([d\xi_i])$. *In particular* $\dim_Q(\ker \sigma_{n-1})^{n+1} = \dim_Q(\ker \rho_A^*)^{n+1}$ *if and only if* $\sigma_{n-1}([d\xi_i])$ *is decomposable for all* $i \in I$.

PROOF. Since $\sigma_{n-1} = \rho_A^*$ for $* \leq n-1$,

$$\sigma([d\theta_j]) = \rho_A^*([d\theta_j]) = 0 \quad \text{and} \quad \sigma_{n-1}([d\xi_i - P_i]) = \sigma_{n-1}([d\xi_i]) - \rho_A(P_i) = 0.$$

Conversely, let $y \in (\ker \sigma_{n-1})^{n+1}$. Since $H^{n+1}(m(A)(n-1))$ is spanned by $(\ker \rho_A^*)^{n+1}$ and $\wedge(x_1, \ldots, x_s)$, we can write y as follows:

$$y = y_1 - y_2, \qquad y_1 \in (\ker \rho_A^*)^{n+1}, \qquad y_2 \in \wedge(x_1, \ldots, x_s)$$

and set $y_1 = \sum_{i \in I} c_i[d\xi_i] + \sum_{j \in J} d_j[d\theta_j]$. Then $0 = \sigma_{n-1}(y) = \sigma_{n-1}(y_1) - \sigma_{n-1}(y_2) = \sum_{i \in I} c_i \sigma_{n-1}([d\xi_i]) - \rho_A^*(y_2)$. Therefore $y_2 = [\sum_{i \in I'} c_i P_i]$, and $c_i = 0$ for $i \notin I'$. Thus we have $y = \sum_{i \in I'} c_i[d\xi_i - P_i] + \sum_{j \in J} d_j[d\theta_j]$. Q.E.D.

Now we consider the following condition.

$(\overline{C_1})$ There exists a G.A. map $\sigma_{n-1} \colon H^*(m(A)(n-1)) \to A^*$ for some integer n such that $\sigma_{n-1} = \rho_A^*$ for $* \leq n-1$ and one of the following is satisfied:

(1) There is ξ_i, $|\xi_i| = n$, such that $\sigma_{n-1}([d\xi_i])$ is not decomposable.

(2) For each D.G.A. automorphism $\chi \colon m(A)(n-1) \to m(A)(n-1)$, there is ξ_i, $|\xi_i| = n$, such that $\rho_A^* \chi^*([d\xi_i - P_i]) \neq 0$.

PROPOSITION 3.4. $(\overline{C_1})$ *is equivalent to* (C_1).

PROOF. By Lemma 3.3, (1) in $(\overline{C_1})$ implies $\dim_Q(\ker \sigma_{n-1})^{n+1} < \dim_Q(\ker \rho_A^*)$ and (2) in $(\overline{C_1})$ implies $\chi^*(\ker \sigma_{n-1}) \not\subset (\ker \rho_A^*)$ for any automorphism χ. Therefore $(\overline{C_1}) \Rightarrow (C_1)$.

If both (1) and (2) in $(\overline{C_1})$ are not satisfied, there is an automorphism χ such that $\rho_A^* \chi^*([d\xi_i - P_i]) = 0$ for all $i \in I'$. This implies $\chi^*(\ker \sigma_{n-1})^{n+1} \subset (\ker \rho_A^*)^{n+1}$ by Lemma 3.3 but since (1) is not satisfied, dimensions of both must coincide by Lemma 3.3. Therefore $\chi^*(\ker \sigma_{n-1})^{n+1} = (\ker \rho_A^*)^{n+1}$ and we have $(C_1) \Rightarrow (\overline{C_1})$. Q.E.D.

4. Examples. We consider some examples. The first three examples are determined by (1) in $(\overline{C_1})$, and Example 4 is determined by (2) in $(\overline{C_1})$.

EXAMPLE 1. Let $A^* = H^*(S^\alpha \vee S^\beta \vee S^\gamma)$, $\alpha, \beta \leqslant \gamma$.

(1.1) α is odd and β is even. Then $A^* \simeq \wedge(a, b)/(ab, b^2)$ for $* \leqslant \gamma$ and $m(A^*) \simeq (a, b, \theta, \theta', \ldots, \xi_{ij}, \ldots, \xi_i, \ldots)$ where $d\theta = ab$, $d\theta' = b^2$, $d\xi_i = d\theta^i$, $d\xi_{ij} = a\xi_i^j, \ldots$. Hence its rational homotopy type is unique if and only if $\gamma - 1 \neq |\xi_1| = 2\alpha + \beta - 2 \neq |\xi_2| = 3\alpha + \beta - 2, \ldots$.

(1.2) Both α and β are odd. Then for $* \leqslant \gamma$, $A^* \simeq \wedge(a, b)/(ab)$, and $m(A^*) \simeq \wedge(a, b, \theta, \xi_a, \xi_b, \ldots)$ where $d\theta = ab$, $d\xi_a = a\theta$, $d\xi_b = b\theta, \ldots$. Hence the type is unique if and only if $\gamma - 1 \neq |\xi_a| = \beta + 2\alpha - 2, \neq |\xi_b| = 2\beta + \alpha - 2, \ldots$.

(1.3) Both α and β are even. Then for $* \leqslant \gamma$, $A^* \simeq \wedge(a, b)/(ab, a^2, b^2)$ and $m(A^*) \simeq \wedge(a, b, \theta_1, \theta_2, \theta_3, \xi_1, \xi_2, \ldots)$ where $d\theta_1 = a^2$, $d\theta_2 = b^2$, $d\theta_3 = ab$, $d\xi_1 = b\theta_1 - a\theta_3$, $d\xi_2 = a\theta_2 - b\theta_3, \ldots$. Hence the type is unique if and only if $\gamma - 1 \neq |\xi_1| = 2\alpha + \beta - 2, \neq |\xi_2| = \alpha + 2\beta - 2, \ldots$.

EXAMPLE 2. $A^* = \wedge(a, b, c)/(ab, b^2, bc) = H^*((S^\alpha \vee S^\gamma) \cup e^{\alpha+\gamma} \vee S^\beta)$, α, γ are odd and β is even. Let $|c| = \gamma = |\xi_1| = 2\alpha + \beta - 3$. Then $ad\xi_1 = a^2\theta = 0$ and $ac \neq 0$ in A^*. Hence there is no G.A. map $\sigma(d\xi_1) = c$, and its rational homotopy type is unique.

EXAMPLE 3. (3.1) Let

$$A_1^* = \wedge(a, b, c, x)/(K, ax, I) \quad \text{and} \quad A_2^* = \wedge(a, b, c, x)/(K, I)$$

where $K = (a^2bc, ab^2c, abc^2)$, $|a| < |b| < |c|$ are even, $|x| = |a| + 2|b| + 2|c| - 1$ and I is the ideal generated by all elements of degree $> |ax|$. Then

$$m(A_i^*)(|x| - 1) = \wedge(a, b, c, \theta_1, \theta_2, \theta_3, \xi_1, \xi_2, \xi_3) \quad (i = 1, 2)$$

where $d\theta_1 = a^2bc$, $d\theta_2 = ab^2c$, $d\theta_3 = abc^2$, $d\xi_1 = c\theta_2 - b\theta_3$, $d\xi_2 = a\theta_3 - c\theta_1$, $d\xi_3 = b\theta_1 - a\theta_2$. Note that $|d\xi_1| > |d\xi_2| > |d\xi_3|$ and $x = |\xi_1| + 1$. Since $ax = 0$ in A_1^* and $a(c\theta_2 - b\theta_3) = -d(b\xi_2 + c\xi_3)$ in $m(A_1^*)(|x| - 1)$, we have a G.A. map

$$\sigma: H^*(m(A_1^*)(|x| - 2)) = H^*(\wedge(a, b, c, \theta_1, \theta_2, \theta_3, \xi_2, \xi_3)) \to A_1^*$$

such that $\sigma(c\theta_2 - b\theta_3) = x$. This clearly satisfies (1) in $(\overline{C_1})$. From the definition of I, $(C_2)_n$ is also satisfied for $n > |x|$. Hence A_1^* has plural homotopy types.

On the other hand since $ax \neq 0$ in A_2^*, we cannot define a G.A. map

$$\sigma: H^*(A_2^*)(|x| - 2) \to A_2^*$$

satisfying $(\overline{C_1})$. By the definition of I, we cannot also define a G.A. map satisfying $(\overline{C_1})$ for higher degree. Hence A_2^* has unique homotopy type. Let

$$(3.2) \quad \begin{aligned} A_3^* &= \wedge\,(a, b, c, x, y, z)/(K, ax + by + cz, I) \quad \text{and} \\ A_4^* &= \wedge\,(a, b, c, x, y, z)/(K, I) \end{aligned}$$

where $|a| = |b| = |c|$ are even, $|x| = |y| = |z| = 5|a| - 1$ and I is as in (3.1).

Since $ad\xi_1 + bd\xi_2 + cd\xi_3 = 0$, we can define a G.A. map

$$\sigma: H^*(m(A_3)(|x| - 2)) = H^*(\wedge\,(a, b, c, \theta_1, \theta_2, \theta_3)) \to A_3$$

such that $\sigma(c\theta_2 - b\theta_3) = x$, $\sigma(a\theta_3 - c\theta_1) = y$, $\sigma(b\theta_1 - a\theta_2) = z$. By the arguments similar to (3.1), we can show that A_3^* has plural homotopy types and A_4^* has unique homotopy type. Let

$$(3.3) \quad A_5^* = \wedge\,(a, b, c, x)/(K, J)$$

where a, b, c are as in (3.1) and $|ab| < |c|$, $|x| = 2|a| + 2|b| + |c| - 1$ and J be the ideal generated by all elements of degree $> |cx|$. Then there is a G.A. map

$$\sigma: H^*(m(A^*)(|x| - 2)) = H^*(\wedge\,(a, b, c, \theta_1, \theta_2)) \to A_5^*$$

such that $\sigma(b\theta_1 - a\theta_2) = x$, therefore (C_1) is satisfied. However, there is no G.A. map

$$\sigma: H^*(m_{|\theta_3|}) = H^*(\wedge\,(a, b, c, \theta_1, \theta_2, \theta_3)) \to A_5^*$$

because $cx + b\sigma(a\theta_3 - c\theta_1) + a\sigma(c\theta_2 - b\theta_3) \neq 0$ in A_5^*. Hence (C_2) is not satisfied. Therefore A_5^* has unique homotopy type.

EXAMPLE 4. When $\sigma_{n-1}(c) = x$ is a ring generator, since $x \notin m(A^*)(n - 1)$, $(\overline{C_1})$ is always satisfied. When x is a decomposable element, it seems complicated whether $(\overline{C_1})$ is satisfied or not.

$$(4.1) \quad A^* = H^*((S^3 \times S^6) \vee S^4) = \wedge\,(a, b, c)/(ab, b^2, bc, c^2),$$
$$|a| = 3, |b| = 4, |c| = 6.$$

Consider the minimal model

$$m(A^*)(8) = \wedge\,(a, b, c, \theta, \theta', \xi)$$

where $ab = d\theta$, $|\theta| = 6$, $b^2 = d\theta'$, $|\theta'| = 7$, $a\theta = d\xi$, $|\xi| = 8$. Then $m(A^*)(|\xi| - 1) \simeq \wedge(a, b, c, \theta, \theta')$, $m(A^*)(7)^9 = Q\{a\theta, ac\}$. There is an automorphism χ for each λ so that $\chi(\theta) = \theta - \lambda c$, $\chi = $ identity for other ring generators, and $\chi(a\theta + \lambda ac) = a\theta = d\xi$. Therefore $(\overline{C_1})$ is not satisfied and A^* has a unique rational homotopy type.

$$(4.2) \quad A^* = H^*(S^3 \vee (S^4 \times S^5)) = \wedge\,(a, b, c)/(ab, b^2, ac),$$
$$|a| = 3, |b| = 4, |c| = 5.$$

Then the minimal model is

$$m(A^*)(7) \simeq \wedge\,(a, b, c, \theta, \theta', \theta'') \quad \text{and} \quad |a| < |b| < |c| < |\theta| < |\theta'| = |\theta''|.$$

Then all automorphisms are $\chi(f) = \lambda_i f$ for all ring generators $f \neq \theta', \theta''$. Hence there is no automorphism so that $\chi(a\theta + \lambda bc) \in dm(A^*)$. Hence $(\overline{C_1})$ is satisfied. Therefore A^* has plural rational types.

5. Uniquely determined cases. In this section we consider some cases when A^* has unique rational homotopy types. From condition $(\overline{C_1})$ the following lemma is immediate.

LEMMA 5.1. *Under the notations* (1) *and* (2) *in* §3, *if* $A^{|\xi_i|+1} = 0$ *for all* ξ_i, *then* A^* *has a unique rational homotopy type.*

Recall that an ideal $I = (a_1,\ldots,a_t)$ is regular if each a_i is not a zero divisor of $\wedge(x_1,\ldots,x_s)/(a_1,\ldots,a_{i-1})$.

THEOREM 5.2 (BODY [1], SULLIVAN [7]). *Let* $A^* = \wedge(x_1,\ldots,x_s)/I$ *and* $I = (a_1,\ldots,a_t)$ *be regular. Then* $m(A^*) \simeq \wedge(x_1,\ldots,x_s,\theta_1,\ldots,\theta_t)$ *where* $d\theta_i = a_i$. *Hence* A *has a unique rational homotopy type.*

PROOF. We shall show that for each degree

$$\ker d \simeq Q\{\wedge(x_1,\ldots,x_s), d\wedge(x_1,\ldots,\theta_t)\} \quad \text{in } \wedge(x_1,\ldots,\theta_t),$$

there is no element ξ in $m(A^*)$ with $d\xi \notin \wedge(x_1,\ldots,x_s)$. Hence then A^* has a unique rational homotopy type from Lemma 5.1.

By induction we assume that

$$\ker d \simeq Q\{\wedge(x_1,\ldots,x_s), d\wedge(x_1,\ldots,\theta_i)\} \quad \text{in } \wedge(x_1,\ldots,\theta_i).$$

Note that $|\theta_j|$ is odd by the regularity. Let $A\theta_{i+1} + B \in \ker d$, with $A, B \in \wedge(x_1,\ldots,\theta_i)$. Then

$$(dA)\theta_{i+1} + Ad\theta_{i+1} + dB = 0.$$

Hence $dA = 0$ and from the assumption we can write $A = dA' + x$ where $x \in \wedge(x_1,\ldots,x_s), A' \in \wedge(x_1,\ldots,\theta_i)$ and moreover $x \notin \text{ideal}(a_1,\ldots,a_i)$ or $x = 0$. Consider

$$0 = d(A\theta_{i+1} + B - d(A'\theta_{i+1})) = d(-x\theta_{i+1} + (B - A'a_{i+1}))$$
$$= -xa_{i+1} + d(B - A'a_{i+1}).$$

But $d(B - Aa_{i+1}) \in d(\wedge(x_1,\ldots,\theta_i)) \subset \text{ideal}(a_1,\ldots,a_i)$. Hence by the regularity, we get $x = 0$. Therefore

$$A\theta_{i+1} + B - d(A'\theta_{i+1}) = B - A'a_{i+1} \in \wedge(x_1,\ldots,\theta_i).$$

By the inductive assumption, we have

$$A\theta_{i+1} + B \in Q\{\wedge(x_1,\ldots,x_s), d\wedge(x_1,\ldots,\theta_{i+1})\}. \quad \text{Q.E.D.}$$

Next we show that the uniqueness is essentially dependent on degree. For an odd prime number p, let $\tilde{A}^{p*}$ be a G.A. obtained from A^* by multiplying each degree by p, namely $\tilde{A}^{p*} \simeq A^*$ if we ignore their degree, and we define $|\tilde{x}| = p|x|$ for corresponding elements x, $\tilde{x}$.

THEOREM 5.4. *Let* A^* *be of finite type, i.e.,* $|A^*| \leqslant n$. *Then if* $p > n$, $\tilde{A}^{p*}$ *has a unique rational homotopy type.*

For the present, we prepare the following lemma.

LEMMA 5.3 [5, 7]. *There is a decomposition $m^i(A) = \bigoplus_{j>0} W_j^i$ such that $dW_j^i \subset W_{j-1}^{i+1}$, $W_j^i \cdot W_m^k \subset W_{j+m}^{i+k}$, where $m^i(A)$ denotes the ith dimensional part of $m^*(A)$.*

PROOF. Suppose we have a decomposition $m^i(A)(n-1) = \bigoplus_{j>0} W_j^i$ such that $dW_j^i \subset W_{j-1}^{i+1}$, $W_j^i \cdot W_m^k \subset W_{j+m}^{i+k}$. Let $m^*(A)(n) = m^*(A)(n-1) \otimes \wedge(\alpha_i, \beta_j)$ $(i = 1,\ldots,s; \; j = 1,\ldots,t)$, where $d\alpha_i = 0$ and $d\beta_j = \sum_{k>0} \gamma_j^k$, $\gamma_j^k \in W_k^{n+1}$. Then $0 = d(d\beta_j) = \sum_{k>0} d\gamma_j^k$, so $d\gamma_j^k = 0$ for each k. Consider the D.G.A. map ρ: $m^*(A)(n) \to A^*$. Since $\rho^*([\gamma_j^k]) = 0$ for $k \geq 1$ and the induced map ρ^* is monomorphic there are η_j^k such that $d\gamma_j^k = \eta_j^k$. Let V_0^n be the subspace spanned by α_i and V_k^n $(k \geq 1)$ be the subspace spanned by η_{k-1}^j $(j = 1,\ldots,t)$. Then by setting $\overline{W}_k^n = W_k^n \oplus V_k^n$ $(k \geq 0)$, we have a desired decomposition using multiplicative law.   Q.E.D.

PROOF OF THEOREM 5.4. Let the minimal model of A^* be denoted by

$$m(A^*) = \wedge(x_1,\ldots,x_s, \eta_1, \eta_2,\ldots),$$

and take each generator η_i to be contained in some W_{k_i} in Lemma 5.3.
We shall construct the minimal model of $\tilde{A}^p{}^*$. Let

$$\tilde{m} = \wedge(\tilde{x}_1,\ldots,\tilde{x}_s, \tilde{\eta}_1, \tilde{\eta}_2,\ldots),$$

$$|\tilde{\eta}_s| = p(i+j) - j \quad \text{if } \eta_s \in W_j^i.$$

If $\alpha = x\eta_{i_1} \cdots \eta_{i_s'}$, $\eta_{i_k} \in W_{j_k}^{i_k}$, then $\alpha \in W_j^i$ where $i = |x| + i_1 + \cdots + i_s$, $j = j_1 + \cdots + j_s$, and

$$|\tilde{\alpha}| = |\tilde{x}\tilde{\eta}_{i_1},\ldots,\tilde{\eta}_{i_s}| = p|x| + p(j_1 + i_1) + \cdots + p(j_s + i_s) - j_s = p(i+j) - j.$$

Hence for all elements $\alpha \in W_j^i$ we have $|\tilde{\alpha}| = p(i+j) - j$.
Let $\tilde{d}\tilde{\eta}_s = \sum \tilde{x}_\beta \tilde{\eta}_\beta$ if $d\eta = \sum x_\beta \eta_\beta$. Since $dW_j^i \subset W_{j-1}^{i+1}$, the degree of $\tilde{d}$ is

$$-(p(i+j) - j) + (p(i+1+j-1) - j + 1) = +1.$$

It is immediate that $H^*(\tilde{m}, \tilde{d}) \simeq \tilde{A}^p{}^*$. Therefore $\tilde{m}$ is the minimal model of $\tilde{A}^p{}^*$.
If $|\tilde{\eta}_s| = pl$ for some l, then $\eta_s \in W_{pl'}^i$ and

$$pl = p(i + pl') - pl' > p(p-1) \geq pn \geq |\tilde{A}^p{}^*|.$$

Therefore from Lemma 5.1 we have the theorem.   Q.E.D.
In the following we see a case when condition (C_2) is satisfied.

THEOREM 5.5. *Under the notations (1), (2) in §2, if there exists $x \neq 0 \in A^{|\xi_i|+1}$ such that the condition $(\overline{C}_1)$ is satisfied and $ax = 0$ for all $a \in A^*$ of positive degree, then A^* has plural rational homotopy types.*

PROOF. We take the generators ξ_i such that each ξ_i is contained in some W_{k_i} in Lemma 4.3.
Let $n = |\xi_i|$. Let $(c, e_1, e_2,\ldots)$ be elements in $m(A^*)(n-1)$ which are ring generators of dimension $\geq n$ in $H^*(m(A^*)(n-1))$. If $[y] = [\zeta]$ in $H^*(m(A^*)(n-1))$ where $y \leq \wedge(x_1,\ldots,x_s)$ and $\zeta \in \text{ideal}(c, e_1, e_2,\ldots)$, then from Lemma 4.3, $y = d(\sum y_i' \theta_i)$ for some $y_i' \in \wedge(x_1,\ldots,x_s)$, and hence $0 = [y] \in H^*(m(A^*)(n-1))$. Since $ax = 0$ for all a of positive degree, we can define a G.A. map   ,

$$\sigma: H^*(m(A^*)(n-1)) \to A^*$$

by $\sigma([c]) = x$ and $\sigma([e_i]) = 0$ for all e_i.

Next assume that for $h \geqslant k \geqslant n - 1$ there is a G.A. map

$$\sigma_k : H^*(m_k) \to A^*$$

such that $\sigma_k([c]) = x$ and each $\sigma_k([e_{k_i}]) = 0$, where $(e_{k_1}, e_{k_2}, \ldots)$ are ring generators of dimension $\geqslant k + 1$ in $H^*(m_k)$.

Let $d\eta_j = y + b$ and $|\eta_j| = l \leqslant h$ where $y \in \wedge(x_1, \ldots, x_s, y_1, \ldots, y_t)$, and $b \in$ ideal$(\theta_1, \ldots, \xi_1, \ldots, \xi_{i-1}, \eta_1, \ldots, \eta_{j-1})$. By the construction η_j in (C) in §3, $\sigma_l([d\eta_j])$ $= \sigma_l([y + b]) = 0$. If $b \neq c$, then by the assumption $\sigma_l([b]) = 0$ and so $\sigma_l([y]) = 0$. Therefore we can make η_i one of the following (1)–(3);

(1) $d\eta_1 = c - x$ (when x is a decomposed element),

(2) $d\eta_i \in \wedge(x_1, \ldots, x_s, y_1, \ldots, y_t)$,

(3) $d\eta_i \in$ ideal$(\theta_1, \ldots, \xi_1, \ldots, \xi_{i-1}, \eta_1, \ldots, \eta_{j-1})$.

We consider the case when x is a decomposed element. Let $(e_1, e_2, \ldots)$ be elements in m_{h+1} which are ring generators of dimension $h + 1$ in $H^*(m_{h+1})$. Let $y \in \wedge(x_1, \ldots, y_k)$, $\zeta \in$ ideal$(c, e_1, e_2, \ldots)$ and let $[y] = [\zeta]$. Let $y - \zeta = d\bar{x}$ and $\bar{x} = \Sigma q_i \eta_1^i$ where $q_i \in \wedge(\ldots, \hat{\eta}_1, \ldots)$. Then we have

$$y - \zeta = dq_0 + q_1 d + dq_1 \eta + \cdots = dq_0 - q_1 x + q_1 c + (dq_1)\eta + \cdots.$$

Considering terms which are contained in $\wedge(x_1, \ldots, y_t)$, we can write

$$y = d\left(\Sigma y_i' \theta_i + \Sigma y_j'' \eta_j^-\right) + q_1' x$$

where y_i, y_i, $d\tilde{\eta}_j \in \wedge(x_1, \ldots, y_t)$ and $q_1' \in \wedge(x_1, \ldots, y_t)$. Therefore we get $[y] =$ $[x][q_1]$. Hence we can define the G.A. map

$$\sigma_{h+1} : H^*(m_{h+1}) \to A^*$$

by $\sigma_{h+1}([c]) = x$ and $\sigma_{h+1}([e_i]) = 0$, since $ax = 0$ in A^*.

When x is a decomposed element, similarly but rather simply we can prove the existence of σ_{h+1}. Therefore the condition (C) in §3 is satisfied.   Q.E.D.

6. Some remarks. Here we note some functorial properties. Let A^*, B^* be G.A. and let $A^* \vee B^* = A^* \otimes B^*/(A^+ \otimes B^+)$ where A^+, B^+ are the ideals of positive elements; namely, if $A^* = H^*(X)$ and $B^* = H^*(Y)$ then $A^* \vee B^* = H^*(X \vee Y)$. The following proposition is immediately obtained by Corollary 3 in [5].

PROPOSITION. (1) *If A^* has plural homotopy types, then so has $A^* \vee B^*$ for any* G.A. *B^*.*

(2) *If A^* has plural homotopy types, then so has $A^* \otimes B^*$ for any* G.A. *B^*.*

The following example shows that subalgebras of a G.A. having unique homotopy type need not have unique homotopy type.

EXAMPLE. Let $A^* = \wedge(a, b, c, x)/(a^2 bc, ab^2 c, abc^2 I)$, $|a| = |b| = |c| = 2$, $|x| = 13$, and I is the ideal generated by elements of degree $\geqslant 14$. Then there is no generator of degree 13 in $m(A)$, so by Lemma 5.1, A^* has unique homotopy type. Consider its subalgebra $B^* = \wedge(a, abc, x)/(a(abc), (abc)^2, I')$, I' is the ideal of elements of degree $\geqslant 14$. Then its minimal model is

$$m(B) = \wedge(a, abc, \theta_1, \theta_2, \xi, \ldots)$$

where $d\theta_1 = a(abc)$, $d\theta_2 = (abc)^2$, $d\xi = (abc)\theta_1 - a\theta_2$. ξ is of degree 12; we can see B^* has plural homotopy types.

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