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FOLIATION PRESERVING LIE GROUP ACTIONS AND CHARACTERISTIC CLASSES

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ABSTRACT. Let $\tilde{\mathfrak{F}}$ be a codimension k foliation of a manifold M and $\mathfrak{F}$ a subfoliation of $\tilde{\mathfrak{F}}$ with codimension q. Let a Lie group G of dimension k act on M transversally locally freely to $\tilde{\mathfrak{F}}$ and preserving $\mathfrak{F}$. Let $\mathfrak{F}'$ be the foliation determined by $\mathfrak{F}$ and the G-action. Then we have the following relations between exotic classes of $\mathfrak{F}$ and $\mathfrak{F}'$: $\alpha_{\mathfrak{F}}([\hat{c}_I c_J]) = \alpha_{\mathfrak{F}'}([\hat{c}_I c_J])$ for $I = (i_1, \ldots, i_\lambda)$, $J = (j_1, \ldots, j_\mu)$, $1 \leq j_\gamma, j_l \leq q - k$, and $\alpha_{\mathfrak{F}}([\hat{c}_I c_J]) = 0$ otherwise.

1. Introduction. Let $\Gamma(\xi)$ denote the set of C^∞-sections of a vector bundle ξ. Let $\mathfrak{F}$ be a C^∞-foliation of a manifold M. We denote by F the subbundle of the tangent bundle $T(M)$, which is determined by $\mathfrak{F}$. It is said that a tangent vector field $Y \in \Gamma(T(M))$ *preserves* $\mathfrak{F}$ if for each $Z \in \Gamma(F)$ we have $[Y, Z] \in \Gamma(F)$. A k-frame field $\{X_1, \ldots, X_k\} \subset \Gamma(T(M))$ is called *transverse to F*, if the span of $X_1, \ldots, X_k$ at each point has 0 intersection with F.

We say that vector fields $X_1, \ldots, X_k$ form a *Lie algebra* mod $\mathfrak{F}$, if there exists C^∞-functions α_{ij}^l and vector fields $Y_{ij} \in \Gamma(F)$, $i, j, l = 1, \ldots, k$, such that

$$[X_i, X_j] = \sum_{l=1}^{k} \alpha_{ij}^l X_l + Y_{ij}.$$

Let ξ_i be the trivial line bundle determined by X_i. If $X_1, \ldots, X_k$ form a Lie algebra mod $\mathfrak{F}$ and each X_i preserves $\mathfrak{F}$, then the subbundle

$$F' = \bigoplus_{i=1}^{k} \xi_i \oplus F \subset T(M)$$

is integrable and defines an extended foliation $\mathfrak{F}'$ of M. Let $\tilde{\mathfrak{F}}$ be a C^∞-foliation of M and $\tilde{F} \subset T(M)$ the subbundle determined by $\tilde{\mathfrak{F}}$. If F is a subbundle of $\tilde{F}$, then $\mathfrak{F}$ is called a *subfoliation* of $\tilde{\mathfrak{F}}$ which is denoted by $\mathfrak{F} \subset \tilde{\mathfrak{F}}$.

Let $\alpha_{\mathfrak{F}}$: $H^*(WO_q) \to H^*(M; \mathbf{R})$ be the map which defines the exotic classes of a foliation $\mathfrak{F}$ of codimension q [**B**]. We assume all manifolds are paracompact Hausdorff C^∞-manifolds without boundary, and all maps and bundles are of class C^∞.

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THEOREM. *Let* $\mathfrak{F}$ *and* $\tilde{\mathfrak{F}}$ *be* C^∞*-foliations of a manifold* M *such that* $\mathfrak{F} \subset \tilde{\mathfrak{F}}$, codim $\mathfrak{F} = q$ *and* codim $\tilde{\mathfrak{F}} = k$. *Let* $\{X_1, \ldots, X_k\}$ *be a* k*-frame field transverse to* $\tilde{\mathfrak{F}}$. *Suppose that each* X_i *preserves* $\mathfrak{F}$ *and* $X_1, \ldots, X_k$ *form a Lie algebra* mod $\mathfrak{F}$. *Let* $\mathfrak{F}'$ *be a foliation determined by* $\{X_1, \ldots, X_k\}$ *and the subbundle* F *corresponding to* $\mathfrak{F}$. *Then we have*

$$\alpha_{\mathfrak{F}}([\hat{c}_I c_J]) = \alpha_{\mathfrak{F}'}([\hat{c}_I c_J]), \quad I = (i_1, \ldots, i_\lambda), J = (j_1, \ldots, j_\mu), 1 \leqslant i_\gamma, j_l \leqslant q - k,$$

and $\alpha_{\mathfrak{F}}([\hat{c}_I c_J]) = 0$ *otherwise.*

This result is motivated by the theorem of Lazarov and Shulman [**LS**]. In §2, we prove our theorem. §3 is devoted to show examples from lift foliations on principal bundles over foliated manifolds.

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2. Proof of the theorem. We denote the transverse (i.e., normal) vector bundles of the foliations $\mathfrak{F}$ and $\mathfrak{F}'$ by $V(\mathfrak{F})$ and $V(\mathfrak{F}')$ respectively. Since F is a subbundle of $\tilde{F}$ and M is paracompact Hausdorff, one can find a subbundle $\tilde{V} \subset \tilde{F}$ such that $\tilde{F} = F \oplus \tilde{V}$. $\{X_1, \ldots, X_k\}$ is a k-frame field transverse to the codimension k foliation $\tilde{\mathfrak{F}}$ and therefore we have

$$T(M) = \bigoplus_{i=1}^{k} \xi_i \oplus F \oplus \tilde{V}.$$

By the definition of transverse vector bundles of foliations, one obtains obviously $V(\mathfrak{F}') = T(M)/F' \cong \tilde{V}$. Let $\tilde{\rho}$: $T(M) \to \tilde{V}$ denote the natural projection map. We set simply $V = V(\mathfrak{F})$ and it follows that $V \cong \oplus_{i=1}^k \xi_i \oplus \tilde{V}$.

Let $\tilde{\nabla}$ be any Bott connection for $\mathfrak{F}'$ on $\tilde{V}$ and ∇' the trivial connection with respect to the global k-frame field $\{X_1, \ldots, X_k\}$ on the trivial bundle $\oplus_{i=1}^k \xi_i$: For any vector field $X \in \Gamma(T(M))$,

$$\nabla'_X(X_i) = 0, \qquad i = 1, \ldots, k.$$

We define a connection ∇ on V by the Whitney sum

$$\nabla = \nabla' \oplus \tilde{\nabla}. \tag{1}$$

Since $\oplus_{i=1}^k \xi_i \oplus \tilde{V}$ is a subbundle of $T(M)$, any section $s \in \Gamma(V)$ can be regarded as an element of $\Gamma(\oplus_{i=1}^k \xi_i \oplus \tilde{V}) \subset \Gamma(T(M))$. Let ρ: $T(M) \to V = T(M)/F$ be the natural projection map. For any vector field $X \in \Gamma(F)$, we have $[X, X_i] \in \Gamma(F)$ and hence

$$\nabla_X(X_i) = 0 = \rho([X, X_i]), \qquad i = 1, \ldots, k.$$

Since any section $s' \in \Gamma(\oplus_{i=1}^k \xi_i)$ is of the form $s' = \sum_{i=1}^k \beta_i X_i$ where β_i is a C^∞-function on M for $i = 1, \ldots, k$, we obtain

$$\nabla_X(s') = \nabla'_X(s') = \rho([X, s']). \tag{2}$$

For any section $\tilde{s} \in \Gamma(\tilde{V})$ and any vector field $X \in \Gamma(F)$, we get

$$\nabla_X(\tilde{s}) = \tilde{\nabla}_X(\tilde{s}) = \tilde{\rho}([X, \tilde{s}]).$$

On the other hand, $\tilde{F} = F \oplus \tilde{V} \subset T(M)$ is integrable and hence the ξ_i-component of the vector field $[X, \tilde{s}]$ at each point is 0. Therefore, for any vector field $X \in \Gamma(F')$, one obtains $\rho([X, \tilde{s}]) = \tilde{\rho}([X, \tilde{s}])$, which means

(3) $$\nabla_X(\tilde{s}) = \rho([X, \tilde{s}]).$$

Since any section $s \in \Gamma(\oplus_{i=1}^k \xi_i \oplus \tilde{V})$ splits uniquely into the sum

$$s = s' \oplus \tilde{s}, \qquad s' \in \Gamma\left(\bigoplus_{i=1}^{k} \xi_i\right) \quad \text{and} \quad \tilde{s} \in \Gamma(\tilde{V}),$$

by the formulas (1), (2) and (3), we have, for any vector field $X \in \Gamma(F)$,

$$\begin{aligned} \nabla_X(s) &= \rho([X, s']) + \rho([X, \tilde{s}]) \\ &= \rho([X, s' + \tilde{s}]) = \rho([X, s]). \end{aligned}$$

Therefore ∇ is a Bott connection for $\mathcal{F}$ on $V \cong \oplus_{i=1}^k \xi_i \oplus \tilde{V}$.

Let $\{\tilde{s}_1, \ldots, \tilde{s}_{q-k}\}$ be a local $(q-k)$-frame section of $\tilde{V}$ and we set $\{s_1, \ldots, s_q\} = \{\tilde{s}_1, \ldots, \tilde{s}_{q-k}, X_1, \ldots, X_k\}$, which is a local q-frame section of V. Let $\tilde{\theta}_{ij}$, $1 \leq i, j \leq q-k$, be the connection forms of $\tilde{\nabla}$ with respect to $\{\tilde{s}_\lambda\}$. Then the connection forms θ_{ij}, $1 \leq i, j \leq q$, of ∇ with respect to $\{s_\mu\}$ are given by the equations

$$\begin{aligned} \theta_{ij} &= \tilde{\theta}_{ij}, \qquad 1 \leq i, j \leq q-k, \\ \theta_{ij} &= 0, \qquad i > q-k \quad \text{or} \quad j > q-k. \end{aligned}$$

In the matrix notation, we have $\theta = \left[\begin{smallmatrix} \tilde{\theta} & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where $\tilde{\theta} = (\tilde{\theta}_{ij})$, $1 \leq i, j \leq q-k$, and $\theta = (\theta_{ij})$, $1 \leq i, j \leq q$.

Let $\tilde{\Omega} = (\tilde{\Omega}_{ij})$ and $\Omega = (\Omega_{ij})$ denote matrices of local curvature forms of $\tilde{\nabla}$ and ∇ respectively. By the above equation on θ and $\tilde{\theta}$, one obtains $\Omega = \left[\begin{smallmatrix} \tilde{\Omega} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Let $I^j(\mathrm{GL}(m; \mathbf{R}))$ denote the vector space of adjoint invariant homogeneous polynomials of degree j on the Lie algebra $\mathfrak{gl}(m; \mathbf{R})$. We identify elements of $I^j(\mathrm{GL}(q-k; \mathbf{R}))$ with those of $I^j(\mathrm{GL}(q; \mathbf{R}))$ by the natural inclusion map $I^j(\mathrm{GL}(q-k; \mathbf{R})) \subset I^j(\mathrm{GL}(q; \mathbf{R}))$. Let $c_j \in I^j(\mathrm{GL}(q; \mathbf{R}))$ be Chern polynomials. (See, e.g., **[B]** or **[C]**.) $c_j(\tilde{\Omega})$ does have meaning for $j \leq q-k$ and we get

(4) $$\begin{aligned} c_j(\Omega) &= c_j(\tilde{\Omega}), \qquad 1 \leq j \leq q-k, \\ c_j(\Omega) &= 0, \qquad j > q-k. \end{aligned}$$

We fix a Riemannian connection $\tilde{\nabla}^0$ on $\tilde{V}$ and then the connection $\nabla^0 = \nabla' \oplus \tilde{\nabla}^0$ on $V = \oplus_{i=1}^k \xi_i \oplus \tilde{V}$ is also a Riemannian connection. Let $\tilde{\theta}^0 = (\tilde{\theta}_{ij}^0)$ be the matrix of local connection forms of $\tilde{\nabla}^0$ with respect to $\{\tilde{s}_\lambda\}$. Then the matrix θ^0 of local connection forms of ∇^0 with respect to $\{s_\mu\}$ is given by $\theta^0 = \left[\begin{smallmatrix} \tilde{\theta}^0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. We form the connection

$$\nabla^* = t\nabla + (1-t)\nabla^0 \qquad (t \in \mathbf{R})$$

on the vector bundle $V \times \mathbf{R} \to M \times \mathbf{R}$. The matrix of connection forms of ∇^* is $\theta^* = t\theta + (1-t)\theta^0$. We denote its curvature matrix by Ω^*. Similarly one obtains

$\tilde{\nabla}^*$, $\tilde{\theta}^*$ and $\tilde{\Omega}^*$ on the vector bundle $\tilde{V} \times \mathbf{R} \to M \times \mathbf{R}$ and we have, for the Chern polynomials $c_j \in I^j(\mathrm{GL}(q;\mathbf{R}))$,

$$c_j(\Omega^*) = c_j(\tilde{\Omega}^*), \quad 1 \leq j \leq q-k,$$
$$c_j(\Omega^*) = 0, \qquad j > q-k,$$

by the same reason with (4). Then by the definition of exotic class of foliation in **[B]**, we have easily

$$\alpha_{\mathfrak{F}}\left(\left[\hat{c}_I c_J\right]\right) = \alpha_{\mathfrak{F}'}\left(\left[\hat{c}_I c_J\right]\right) \quad \text{for } 1 \leq i_\gamma, j_\mu \leq q-k,$$

and $\alpha_{\mathfrak{F}}([\hat{c}_I c_J]) = 0$ otherwise, which proves our theorem.

REMARK. The Roussarie foliation (see, e.g., **[B]**) shows that, given $\mathfrak{F}$ and $X_1,\ldots,X_k$, in general, $\nabla' \oplus \tilde{\nabla}$ is not a Bott connection and is not even $J(>q)$-homotopic to it in the sense of **[L]**. The existence of $\tilde{\mathfrak{F}}$ makes $\nabla' \oplus \tilde{\nabla}$ a Bott connection.

3. Examples. Let N be a manifold and $\mathfrak{F}_N$ a C^∞-foliation of codimension q on N. We denote by F_N the subbundle of $T(N)$ which is determined by $\mathfrak{F}_N$. Let G be a Lie group, $G_0 \subset G$ a discrete subgroup and $\pi\colon E \to N$ a principal G-bundle, the structure group of which has G_0-reduction. Then there exists a homomorphism h of the fundamental group $\pi_1(N)$ to G_0. Therefore $\pi_1(N)$ acts on G by the left multiplication via h and at the same time it acts on the universal covering manifold $\overline{N}$ of N by covering transformation. It is well known that $E \cong \overline{N} \times_{\pi_1(N)} G$. Since the diagonal action of $\pi_1(N)$ on $\overline{N} \times G$ preserves the foliation $\{\overline{N} \times \{g\}\}_{g \in G}$, it gives rise to a codimension $k = \dim G$ foliation $\tilde{\mathfrak{F}}$ of E. Obviously $\tilde{\mathfrak{F}}$ is invariant under the right action of G. Moreover $\mathfrak{F}_N$ determines a foliation $\overline{\mathfrak{F}}_N = \{\bar{\mathfrak{L}}\}$ of codimension q of $\overline{N}$, which is preserved by the covering transformation of $\pi_1(N)$. Hence the diagonal action of $\pi_1(N)$ on $\overline{N} \times G$ preserves the foliation $\{\bar{\mathfrak{L}} \times \{g\}\}_{\bar{\mathfrak{L}} \in \overline{\mathfrak{F}}_N, g \in G}$ of codimension $q + \dim G$. Therefore one obtains a codimension $q + \dim G$ foliation $\mathfrak{F}$ of E. $\mathfrak{F}$ is also invariant under the right action of G. Since $F = T(\mathfrak{F})$ is a subbundle of $\tilde{F} = T(\tilde{\mathfrak{F}})$, we have $\mathfrak{F} \subset \tilde{\mathfrak{F}}$.

In our theorem, we take E for M and, at each point of E, take images of the basis vectors of the Lie algebra $\mathfrak{G}$ of G under the right action of G, for $X_1,\ldots,X_k$, $k = \dim \mathfrak{G}$. The k-frame field $\{X_1,\ldots,X_k\}$ is obviously transverse to $\tilde{\mathfrak{F}}$ and each X_i, $i = 1,\ldots,k$, preserves $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$, because $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$ are invariant under the right action of G. Moreover we have $\operatorname{codim} \tilde{\mathfrak{F}} = \dim G = k$. Therefore it follows from our theorem that

$$\alpha_{\mathfrak{F}}\left(\left[\hat{c}_I c_J\right]\right) = \begin{cases} \alpha_{\mathfrak{F}'}\left(\left[\hat{c}_I c_J\right]\right) & \text{for } 1 \leq i_\gamma, j_\mu \leq q, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathfrak{F}'$ is the extended foliation of $\mathfrak{F}$ by $X_1,\ldots,X_k$. Since π is a projection, π is transverse to $\mathfrak{F}_N$. Let $\pi^*(\mathfrak{F}_N)$ denote the pullback of $\mathfrak{F}_N$ by π and π_* denote the differential map of π. Then we have $\pi^*(\mathfrak{F}_N) = \mathfrak{F}'$. For a Bott connection and a Riemannian connection in the definition of $\alpha_{\mathfrak{F}'}([\hat{c}_I c_J])$, one can take pullbacks by π of those in the definition of $\alpha_{\mathfrak{F}_N}([\hat{c}_I c_J])$. Therefore we have

$$\alpha_{\mathfrak{F}'}\left(\left[\hat{c}_I c_J\right]\right) = \pi^* \alpha_{\mathfrak{F}_N}\left(\left[\hat{c}_I c_J\right]\right),$$

and hence

$$\alpha_{\mathfrak{F}}([\hat{c}_I c_J]) = \begin{cases} \pi^* \alpha_{\mathfrak{F}_N}([\hat{c}_I c_J]) & \text{for } 1 \leqslant i_\gamma, j_\mu \leqslant q, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $\alpha_{\mathfrak{F}_N}([\hat{c}_I c_J]) = 0$ then one obtains $\alpha_{\mathfrak{F}}([\hat{c}_I c_J]) = 0$. Taking the one leaf foliation of N for $\mathfrak{F}_N$, obviously one gets $\mathfrak{F} = \tilde{\mathfrak{F}}$ and $\alpha_{\tilde{\mathfrak{F}}}([\hat{c}_I c_J]) = 0$.

Let f, g be orientation preserving diffeomorphisms of S^1 such that $f \circ g = g \circ f$. We denote the group of integers by $\mathbf{Z}$ and define an action of $\mathbf{Z} \oplus \mathbf{Z}$ without fixed points on $V = \mathbf{R}^2 \times S^1$ by

$$\tilde{f}(x_1, x_2, \theta) = (x_1 + 1, x_2, f(\theta)),$$
$$\tilde{g}(x_1, x_2, \theta) = (x_1, x_2 + 1, g(\theta)).$$

The codimension 1 foliation of V given by $\theta =$ const. induces a codimension 1 foliation $\mathfrak{F}_{T^3}$ of the quotient manifold $V/(\mathbf{Z} \oplus \mathbf{Z}) \cong T$ (3 dimensional torus). M. R. Herman [H] has shown that the Godbillon-Vey class of $\mathfrak{F}_{T^3}$ is 0:

$$\mathfrak{G}_v(\mathfrak{F}_{T^3}) = \alpha_{\mathfrak{F}_{T^3}}([\hat{c}_I c_J]) = 0.$$

We set $G_0 = \{[\begin{smallmatrix} \pm 1 & 0 \\ 0 & 1 \end{smallmatrix}]\} \cong \mathbf{Z}_2$ (the group of integers mod 2). Then G_0 is a subgroup of $G = \mathrm{GL}(2, \mathbf{R})$. A nontrivial representation $\mathbf{Z} \oplus \mathbf{Z} \to G_0 \cong \mathbf{Z}_2$ defines a left action of $\mathbf{Z} \oplus \mathbf{Z}$ on $\mathrm{GL}(2, \mathbf{R})$. Then one obtains a principal G-bundle

$$V \times_{\mathbf{Z} \oplus \mathbf{Z}} G = E \to T^3 = N,$$

the structure group of which has $G_0 \cong \mathbf{Z}_2$ reduction. The 2 dimensional foliation $\mathfrak{F}_{T^3}$ gives rise to the 2 dimensional foliation $\mathfrak{F}$ of $V \times_{\mathbf{Z} \oplus \mathbf{Z}} G$ stated in the beginning of this section. By our theorem, we obtain $\mathfrak{G}_v(\mathfrak{F}) = 0$.

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