THE ONLY GENUS ZERO \( n \)-MANIFOLD IS \( S^n \)

MASSIMO FERRI AND CARLO GAGLIARDI

Abstract. All \( n \)-manifolds of regular genus zero, i.e. admitting a crystallization which regularly imbeds into \( S^2 \), are proved to be homeomorphic to \( S^n \). A conjecture implying the Poincaré Conjecture in dimension four is also formulated.

Sunto. Si dimostra che tutte le \( n \)-varietà di genere regolare zero, cioè aventi una cristallizzazione che si immerge regolarmente in \( S^2 \), sono omeomorfe a \( S^n \). Si formula anche una congettura che implica quella di Poincaré in dimensione quattro.

1. Throughout this paper, we work in the PL category, for which we refer to [RS]; for graph theory, we refer to [Har]. \( \cong \) denotes PL-homeomorphism.

An \( h \)-coloured graph \((\Gamma, \gamma)\) is a multigraph \( \Gamma \), regular of degree \( h \), together with a coloration \( \gamma \) of the edges by \( h \) colours. If \( \mathcal{C} \) is the colour set, and \( \mathcal{B} \subset \mathcal{C} \), \( \Gamma_{\mathcal{B}} \) will denote the subgraph of \( \Gamma \) generated by the edges \( e \) such that \( \gamma(e) \in \mathcal{B} \). Given a colour \( c \in \mathcal{C} \), \( \hat{c} \) will denote the set \( \mathcal{C} - \{c\} \). An \( h \)-coloured graph \((\Gamma, \gamma)\) is said to be contracted if \( \Gamma_{c} \) is connected for each \( c \in \mathcal{C} \).

To every \((n + 1)\)-coloured graph \((\Gamma, \gamma)\), there corresponds an \( n \)-dimensional pseudocomplex \( K(\Gamma) \), whose \( i \)-simplexes are in one-one correspondence with the connected components of the subgraphs \( \Gamma_{\mathcal{B}} \) for all colour subsets \( \mathcal{B} \) of cardinality \( \#\mathcal{B} = n - i \). Note that, if \((\Gamma, \gamma)\) is contracted, then \( K(\Gamma) \) has exactly \( n + 1 \) vertices. For every closed, connected \( n \)-manifold \( M \), there exists at least one contracted \((n + 1)\)-coloured graph \((\Gamma, \gamma)\) such that \( |K(\Gamma)| \simeq M \); such a graph is called a crystallization of \( M \), and \( K(\Gamma) \) a contracted triangulation of \( M \). For the existence and equivalence theorems for crystallizations, see [P, F, FG]; these and other results are also summarized in [FGG].

We recall the notion of regular genus of a manifold, defined in [G3], which generalizes the genus of a surface and Heegaard genus of a 3-manifold. A 2-cell imbedding [Wh, p. 40] \( i: |\Gamma| \to F \) of an \((n + 1)\)-coloured graph \((\Gamma, \gamma)\) into a closed surface \( F \) is said to be regular if there exists a cyclic permutation \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_n) \) of the colour set, such that each region of \( i \) is bounded by the image of a cycle, whose edges are alternatively coloured by \( \varepsilon_i, \varepsilon_{i+1} \) (\( i \) being an integer mod \( n + 1 \)). The regular genus \( \rho(\Gamma) \) of \((\Gamma, \gamma)\) is defined to be the least genus of a surface into which
(\Gamma, \gamma) regularly imbeds. Given a closed n-manifold \(M\), its regular genus (or simply genus) \(\mathcal{G}(M)\) is defined as the integer

\[ \mathcal{G}(M) = \min \{ \rho(\Gamma) | (\Gamma, \gamma) \text{ is a crystallization of } M \}. \]

As usual, we shall identify a graph with its imbedded image.

[G3, Corollary 7] asserts, among other things, that a 4-manifold of genus zero is simply-connected. We shall extend this result to dimension \(n\). This permits us to compute \(\mathcal{G}(S^1 \times S^n)\), and further to prove the following fact, which confirms the geometrical significance of this invariant.

**Theorem 1.** Let \(M\) be a closed, connected n-manifold; then

\[ \mathcal{G}(M) = 0 \Rightarrow M \approx S^n. \]

**Remark 1.** In view of Theorem 1, it would be interesting to study the behaviour of \(\mathcal{G}\) with respect to connected sums. \(\mathcal{G}\) is easily proved to be subadditive by direct construction. It is trivially additive in dimension 2; in dimension 3, the Heegaard genus—hence also the regular genus—is known to be additive too [Hak, §7]. If the same property held in dimension 4, as we conjecture, this would imply an affirmative answer to the 4-dimensional Poincaré Conjecture. In fact, as it is well known [M, §1.1; Wa; C], if \(M\) is a 4-dimensional homotopy sphere then, for a suitable nonnegative integer \(k\), \(M \# k(S^2 \times S^2) \approx S^4 \# k(S^2 \times S^2)\). But this would imply that \(\mathcal{G}(M) = 0\), whence \(M \approx S^4\).

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2. From now on, \(\Delta_n = \{i \in \mathbb{Z} | 0 \leq i \leq n\}\) will be assumed as a colour set. For each \(\mathfrak{B} \subseteq \Delta_n\), \(\mathfrak{B}_{\mathfrak{B}}\) will denote the number of connected components of \(\Gamma_{\mathfrak{B}}\).

**Lemma 1.** Let \((\Gamma, \gamma)\) be a contracted \((n+1)\)-coloured graph, such that \(\rho(\Gamma) = 0\), and \(\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)\), a cyclic permutation of \(\Delta_n\) associated to a regular imbedding \(\iota\) of \((\Gamma, \gamma)\) into \(S^2\). Let \(\mathfrak{B} \subseteq \Delta_n\) contain at least three colours \(\varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}\) consecutive in \(\varepsilon\) (\(i\) taken in \(\mathbb{Z}_{n+1}\)). Then \(\mathfrak{B}_{\mathfrak{B}} = \mathfrak{B}_{\mathfrak{B}(\varepsilon_i)}\).

**Proof.** As \((\Gamma, \gamma)\) is contracted, \(\Gamma_{\mathfrak{B}}\) is connected. Call \(\gamma'\) and \(\iota'\) the restrictions of \(\gamma\) and \(\iota\) respectively to the latter graph; then \((\Gamma_{\mathfrak{B}}, \gamma')\) is an \(n\)-coloured graph, regularly imbedded by \(\iota'\) into \(S^2\). Namely, \(\iota'\) is a 2-cell imbedding [Wh, Theorem 6.11], and colours \(\varepsilon_{i-1}, \varepsilon_{i+1}\) are now contiguous in the corresponding permutation of \(\Delta_n - \{\varepsilon_i\}\); hence, \((\varepsilon_{i-1}, \varepsilon_{i+1})\)-coloured cycles bound regions of \(\iota'\).

Therefore, each edge coloured by \(\varepsilon_i\) joins two vertices of the same component of \(\Gamma_{\{i-1, i+1\}}\), thus also of the same component of \(\Gamma_{\mathfrak{B}(\varepsilon_i)}\). \(\square\)

**Lemma 2.** Let \((\Gamma, \gamma)\) and \(\varepsilon\) be as in Lemma 1. Let further \(\mathfrak{B} = \Delta_n - \mathfrak{B}'\), where \(\mathfrak{B}'\) contains no two colours consecutive in \(\varepsilon\). Then \(\mathfrak{B}_{\mathfrak{B}} = 1\).

**Proof.** Follows from Lemma 1, by induction on \(\# \mathfrak{B}'\). \(\square\)

**Proposition 1.** For a closed, connected n-manifold \(M\), \(\mathcal{G}(M) = 0 \Rightarrow M\) is simply-connected.
Proof. Obvious for \( n = 2 \). For \( n > 2 \), if \((\Gamma, \gamma)\) of Lemma 2 is a crystallization of \( M \), and \( \mathcal{S} = \Delta_n - \{i, j\} \) with \( i \) and \( j \) not consecutive in \( \varepsilon \), then there is only one component of \( \Gamma_{\mathcal{S}} \). Then [G2, §6, Proposition 9] proves the statement. □

As conjectured in [FG2, §6], we have

**Corollary 1.** \( \mathcal{G}(S^1 \times S^n) = 1 \).

**Proof.** \( \mathcal{G}(S^1 \times S^n) > 0 \) by Proposition 1.

In order to see that \( \mathcal{G}(S^1 \times S^n) \leq 1 \), consider the following construction of a crystallization of \( S^1 \times S^n \), which generalizes [G2, Figures 1,8] [FG2, Figures 4,7] and is obtained by applying the method illustrated in [FG2, §2].

Take \( 2n + 4 \) vertices \( v_j^i (i \in \Delta_1, j \in \Delta_{n+1}) \). Join \( v_j^i \) with \( v_{j+1}^i \) by an edge coloured by \( j \). Put a further edge coloured by \( n + 1 \) between \( v_0^i \) and \( v_{n+1}^i \) if \( n \) is even, between \( v_0^i \) and \( v_{n+1}^i \) if \( n \) is odd. Finally, join \( v_0^i \) with \( v_j^i \) by \( n \) edges coloured by the \( n \) colours not yet used around those vertices.

The fact that such a graph can be regularly imbedded into the torus—with respect to every cyclic permutation of \( \Delta_{n+1} \)—follows from the equality \( \mathcal{G}(i, j) = n \) for all \( i, j \in \Delta_{n+1}, i \neq j \) (see [FGG, §5]). □

3. Proof of Theorem 1. It is trivial to see that \( M = S^n \Rightarrow \mathcal{G}(M) = 0 \), as \( S^n \) admits a standard crystallization consisting of two vertices joined by \( n + 1 \) differently coloured edges; this graph obviously imbeds regularly into \( S^2 \) with respect to every cyclic permutation of \( \Delta_n \).

The proof of the converse implication consists of some general considerations followed by three parts, relative to the cases (A) \( n \) odd, (B) \( n \) even and \( n = 4 \), (C) \( n = 4 \).

In the following construction, which was first introduced in [G1], \( M \) is an arbitrary closed \( n \)-manifold (not necessarily of genus zero), \((\Gamma, \gamma)\) a given crystallization of it, and \( K \) the relative contracted triangulation.

In the vertex set \( V = \{v_0, \ldots, v_n\} \) of \( K \), assume that \( v_i \) corresponds to \( \Gamma_i \). For each nonvoid subset \( W \) of \( V \), set \( W' = V - W \), and call \( K_W \) the contracted subcomplex of \( K \) generated by \( W \). If \( W = h + 1 \), then \( \dim K_W = h \). Furthermore, if \( \mathcal{S} \) is the subset of \( \Delta_n \) such that \( W = \{v_i \mid i \in \mathcal{S}\} \) and \( \mathcal{S}' = \Delta_n - \mathcal{S} \), then the number of \( h \)-simplexes of \( K_W \) equals \( \mathcal{G}(i, j) \); this is easy to check. Now let \( L \) be the largest subcomplex of \( S^d K \), disjoint from \( S^d K_W \cup S^d K_{W'} \). Then \( L \), whose space is a closed \( (n - 1) \)-manifold, splits \( K \) into two complementary subcomplexes, \( N_W \) and \( N_{W'} \) say, having \( L \) as common boundary. Moreover, \( \mid N_W \mid \) and \( \mid N_{W'} \mid \) are regular neighbourhoods, in \( |K| \), of \( |K_W| \) and \( |K_{W'}| \) respectively. Observe that, in dimension three, if \( \mathcal{G}(W) = 2 \), then \(|N_W|, |N_{W'}|\) is a Heegaard splitting of \( M \).

From now on, the hypothesis \( \rho(\Gamma) = 0 \) will be assumed, and \( \iota: |\Gamma| \rightarrow S^2 \) will denote a regular imbedding of \((\Gamma, \gamma)\); w.l.o.g., \( \iota \) can be assumed to be associated to the fundamental cyclic permutation \( \epsilon = (0, 1, \ldots, n) \).

\( ^2 \text{Sd means "barycentric subdivision of";} \) it carries every pseudocomplex to a simplicial complex.
(A) $n = 2r + 1$, $r \geq 0$.

Set $\mathcal{B} = \{2k + 1 \mid 0 \leq k \leq r\}$, $\mathcal{B}' = \Delta_n - \mathcal{B}$; call $W, W'$ the corresponding sub-
sets of $V$. By Lemma 2, $g_{\mathcal{B}} = g_{\mathcal{B}'} = 1$, whence $K_w$ and $K_{w'}$ consist of exactly one
$r$-simplex each. Therefore $|N_w|$ and $|N_{w'}|$ are closed $(2r + 1)$-balls; they cover $M$, and meet in their common boundary $|L|$. Thus $M \cong S^{2r+1}$.

(B) $n = 2r$, $r \neq 2$.

$\mathcal{B}, \mathcal{B}', W, W'$ as in case (A). Here, Lemma 2 only assures that $g_{\mathcal{B}'} = 1$, hence that
$|N_{w'}|$ is a $2r$-ball. The $2r$-complex $N_{w'}$, whose boundary $L$ has a $(2r - 1)$-sphere as
space, has the homotopy type of the $(r - 1)$-complex $K_{w'}$. These facts, applied to
the Mayer-Vietoris homology sequence of $K = K_w \cup K_{w'}$ and $L = K_w \cap K_{w'}$, together with Poincaré duality, imply that $M \cong |K|$ is a homology sphere. Therefore,
as a consequence of Proposition 1 and of the Hurewicz isomorphism theorem, $M$ is
even a homotopy sphere. This, which holds for all $r$, implies that $M \cong S^{2r}$ when
$r \neq 2$, by the generalized Poincaré Conjecture (Smale, Stallings and Zeeman).

(C) $n = 4$.

$\mathcal{B} = \{1, 3\}$, $\mathcal{B}' = \{0, 2, 4\}$; $W, W'$ as before. Again, $g_{\mathcal{B}'} = 1$ implies that $|N_{w'}|$ is a
4-ball.

In order to show that $|N_{w'}|$ is a 4-ball too, let us examine $K_{w'}$ in some detail.
Since $g_{(1,3,4)} = g_{(0,1,3)} = 1$ by Lemma 2, $K_{(v_0, v_2)}$ and $K_{(v_0, v_4)}$ are formed by one
1-simplex each. Hence all triangles forming $K_{w'}$ have two edges in common; then
$K_{w'}$ will be a cone over the 1-pseudocomplex $K_{(v_0, v_4)}$ if it consists of as many
triangles as there are edges in $K_{(v_0, v_4)}$. But this is actually the case, as $g_{(1,2,3)} = g_{(1,3)}$
by Lemma 1. Therefore $|K_{w'}|$ is collapsible, $|N_{w'}|$ is a 4-ball (by Whitehead’s
theorem [RS, Corollary 3.27]), and $M \cong S^4$. □

For $n \geq 2$ we have

**Corollary 2.** Let $(\Gamma, \gamma)$ be a contracted $(n + 1)$-coloured graph such that
$\rho(\Gamma_i) = 0$ for each $i \in \Delta_n$. Then $|K(\Gamma)|$ is a manifold.

**Proof.** For each $i \in \Delta_n$, $\Gamma_i$ is connected and of regular genus zero. If $n = 2$, $\Gamma_i$
is a cycle and hence represents $S^1$. If $n \geq 3$, the fact that $|K(\Gamma_i)| \cong S^{n-1}$ is assured
by Corollary 3. If $n \geq 3$. This proves that, for each vertex $v$ of $K(\Gamma)$, $|K(v, Sd K(\Gamma))| \cong S^{n-1}$, and this suffices to prove the statement (compare [F, Proposition 16]). □

**Corollary 3.** Let $(\Gamma, \gamma)$ be a connected $(n + 1)$-coloured graph such that $\rho(\Gamma) = 0$. Then $|K(\Gamma)| \cong S^n$.

**Proof.** By eliminating a suitable number of dipoles of type 1 [FG, §3] one obtains a contracted graph $(\Gamma', \gamma')$. Now let $e: |\Gamma| \to S^2$ be a regular imbedding of
$(\Gamma, \gamma)$ into $S^2$ relative to the cyclic permutation $e$. Then by [FG, Lemma 1] there
exists also an imbedding $e': |\Gamma'| \to S^2$ relative to the same $e$.

If $|K(\Gamma')|$ is a manifold, i.e. if $(\Gamma', \gamma')$ is a crystallization, then $|K(\Gamma')| \cong |K(\Gamma)|$.
But $|K(\Gamma')|$ is actually a manifold by Corollary 2, since $e'$ induces a regular
imbedding of each $(\Gamma', \gamma|_{\Gamma'})$ into $S^2$. Therefore $|K(\Gamma')| \cong |K(\Gamma')| \cong S^n$ by Theorem
1 applied to $(\Gamma', \gamma')$. □
REFERENCES


ISTITUTO DI MATEMATICA, FACOLTÀ DI INGEGNERIA, V. CLAUDIO, 21, I 80125 NAPOLI, ITALIA