THE DIMENSION OF INVERSE LIMIT AND $N$-COMPACT SPACES

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ABSTRACT. For each $k = 1, 2, \ldots, \infty$, we construct a normal $N$-compact
space $X$ with $\dim X = k$, where $\dim$ denotes covering dimension, which is
the limit space of a sequence of zero-dimensional Lindelöf spaces.

Let $X$ be the limit space of an inverse sequence $(X_n, f_{nm})$. In [1], we showed that
$\dim X$ can be positive even if $X$ is normal and $X_n$ is Lindelöf and zero dimensional
for each $n$ in $N$, the set of natural numbers. In this paper we continue investigating
the behaviour of covering dimension under inverse limits. We generalise the con-
struction in [1] to obtain, for each $k = 1, 2, \ldots, \infty$, an inverse sequence $(X_n, f_{nm})$
of zero-dimensional Lindelöf spaces with limit space $X$ such that $X$ is normal and
$\dim X = \text{Ind} X = k$, where Ind denotes large inductive dimension. The space $X$
is, in addition, first countable, locally compact, countably paracompact and collectionwise normal. Recall that a space is called $N$-compact if it is the inverse limit
of countable discrete spaces. Every zero-dimensional Lindelöf space is $N$-compact,
and so is the inverse limit of $N$-compact spaces. It follows that $X$ is $N$-compact.$N$-compact spaces of positive covering dimension have previously been constructed
in [5, 6] and also [7, 8], but our space $X$ seems to be the first example showing that
$N$-compact spaces can have infinite dimension.1

In this paper, all spaces are Tychonoff. The usual metric on the Cantor set $C$
is denoted by $d$. $I$ denotes the unit interval, $\omega(c)$ the first ordinal of cardinality $c$, the
cardinality of the continuum, and $|X|$ the cardinality of a set $X$.

For standard results in Dimension Theory we refer to [4].

1. Preliminaries. The construction of the spaces $X$ and $X_n$ in this paper is only
slightly more complicated than that of the corresponding spaces in [1]. In both
papers, we draw from the techniques employed in [2, 7, 8]. The constructions in
[1] made use of Wage's complete separable metric $\rho$ on $C$ which has the following
properties: (a) the $\rho$-topology is finer than the usual topology on $C$, (b) every $\rho$-Borel
set of $C$ is $d$-Borel, and (c) every $\rho$-open set disjoint from a certain fixed $\rho$-closed
set $E$ has boundary of cardinality $c$. The constructions in the present paper are
based on the existence of a separable metric $e$ on $C$ with the properties enumerated
in the following result.

PROPOSITION 1. For each $k = 1, 2, \ldots, \infty$, there exists a separable metric $e$ on
$C$ with $d \leq e$ and $k$ pairs of disjoint $e$-closed sets $E_i, F_i, i = 1, 2, \ldots, k$, such that

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1The referee has informed the author that R. Engelking and E. Pol have recently constructed,
for each $k \in \{1, 2, \ldots, \infty\}$ a Lindelöf, zero-dimensional space $X = X(k)$ such that $\dim X^2 = k$. © 1982 American Mathematical Society
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(1) every uncountable e-closed and every nonempty e-open subset of C has cardinality c.

(2) Whenever $L_i, i = 1, \ldots, k$, is an e-partition between $E_i$ and $F_i$, then

$$\bigcap_{i=1}^{k} L_i = c.$$ 

(3) $\dim(C, e) = k$.

N.B. The condition $d < e$ implies that the e-topology is finer than the usual topology on C.

PROOF. Let $Y = C \times I^k$ and $\pi = Y \to C$ the canonical projection. Let $\mathcal{G}$ be the collection of all subsets $G$ of $Y$ such that $|\pi(G)| = c$ and $G$ is either open or closed. Note that $|\mathcal{G}| = c$ and we may choose an enumeration $\{G_\alpha : \alpha < \omega(c)\}$ of $\mathcal{G}$ such that for each $G$ in $\mathcal{G}$, $G = G_\alpha$ for $c$ ordinals $\alpha < \omega(c)$. For each $\alpha < \omega(c)$, since $|\pi(G_\alpha)| = c$, we can choose by transfinite induction, a point $x_\alpha$ in $\pi(G_\alpha)$ so that $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$. Next, we define a function $f : C \to I^k$ as follows. If $x = x_\alpha$ for some $\alpha < \omega(c)$, we choose $f(x)$ so that $(x, f(x))$ is in $G_\alpha$. If not, we set $f(x) = 0$.

Let $X$ be the subspace $\{(x, f(x)) : x \in C\}$ of $Y$. Then $\pi : X \to C$ is bijective and continuous. Consider an uncountable closed set $E$ of $X$. Then $E = F \cap X$ for some closed subset $F$ of $Y$. Now the closed set $\pi(F)$ of $Y$ is uncountable and hence has cardinality $c$, so that $F$ is in $\mathcal{G}$. If $A = \{\alpha < \omega(c) : F = G_\alpha\}$, then $E$ contains $\{(x_\alpha, f(x) : \alpha \in A\}$, and since $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, and $|A| = c$, we have $|E| = c$. Similarly, every nonempty open subset of $X$ has cardinality $c$. In fact, $|G \cap X| = c$ for every $G$ in $\mathcal{G}$.

Let $A_i, B_i, i = 1, \ldots, k$, be the pairs of opposite faces of $I^k$. Let $U_i, V_i$ be open sets of $Y$ such that $C \times A_i \subset U_i, C \times B_i \subset V_i$ and $U_i \cap V_i = \emptyset$. Set $E_i = X \cap U_i, F_i = X \cap V_i, i = 1, \ldots, k$, and suppose $L_i$ is a partition in $X$ between $E_i$ and $F_i$. Then there exist disjoint open sets $G_i, H_i$ of $Y$ with $E_i \subset G_i, F_i \subset H_i$ and $X - (G_i \cup H_i) \subset L_i$. Let $P_i = U_i \cup (G_i - V_i), Q_i = V_i \cup (H_i - U_i)$ and $M_i = Y - (P_i \cup Q_i)$. Then $M_i$ is a partition in $Y$ between $C \times A_i$ and $C \times B_i$ with $M_i \cap X \subset L_i$. It follows that $\bigcap_{i=1}^{k} M_i$ contains at least one point from $\{x\} \times I^k$ for each $x$ in $C$. Thus $|\pi(\bigcap_{i=1}^{k} M_i)| = c$, which implies that

$$\bigcap_{i=1}^{k} M_i \cap X = \bigcap_{i=1}^{k} L_i = c.$$ 

Now, by the Otto-Eilenberg characterisation of covering dimension [4, Theorem 1.7.9], $\dim X \geq k$ and, since $X$ is a subspace of the $k$-dimensional space $Y$, we have $\dim X = k$.

Finally, to complete the proof of Proposition 1, it suffices to let, for $x, y$ in $C$, $e(x, y)$ denote the usual Euclidean distance between the points $(x, f(x))$ and $(y, f(y))$.

We shall need the following result, which is the analogue for the metric $e$ on $C$ of the well-known theorem on the existence of Bernstein sets in a complete separable metric space.

PROPOSITION 2. There is a partition $A_1, A_2, A_3, \ldots$ of $C$ such that $|A_i \cap F| = c$ for each $i$ in $N$ and each uncountable e-closed set $F$ of $C$. 
PROOF. Let $\mathcal{F}$ be the collection of all uncountable $e$-closed sets of $C$. Let 
$\{F_\alpha : \alpha < \omega(c)\}$ be an enumeration of $\mathcal{F}$ such that for each $F$ in $\mathcal{F}$, $F = F_\alpha$ for 
countable ordinals $\alpha < \omega(c)$. For each $\alpha$, by Proposition 1, $|F_\alpha| = c$ and thus we can
choose by transfinite induction points $x_{\alpha 1}, x_{\alpha 2}, \ldots$ in $F_\alpha$ so that $x_{\alpha n} \neq x_{\beta m}$ if
$(\alpha, n) \neq (\beta, m)$. It now suffices to let $A_i = \{x_{\alpha i} : \alpha < \omega(c)\}$ for $i = 2, 3, \ldots$, and
$A_1 = C - \bigcup_{i=2}^\infty A_i$.

N.B. It follows from Proposition 1(1), that each $A_i$ is $e$-dense in $C$.

2. The construction. In the sequel, $k$ denotes a fixed number of the set
$\{1, 2, \ldots, \omega(c)\}$, $e$ the metric on $C$ given by Proposition 1, and $A_1, A_2, \ldots$ the partition
of $C$ provided by Proposition 2.

Let $\{(S_{\alpha 1}, S_{\alpha 2}, \ldots) : \alpha < \omega(c)\}$ be the collection of all sequences of countable
subsets of $A_1$ with $|\bigcap_{i=1}^\infty S_{\alpha i}| = c$. Since
$$|A_1 \cap \bigcap_{i=1}^\infty S_{\alpha i}| = c,$$
for each $\alpha < \omega(c)$, we can choose $x_{\alpha}$ in $A_1 \cap \bigcap_{i=1}^\infty S_{\alpha i}$ and a sequence $\{x_{\alpha n}\}$ in
$A_1$ so that $e(x_{\alpha}, x_{\alpha n}) < \frac{1}{n}$, $\{x_{\alpha n}\}$ contains infinitely many points from each $S_{\alpha i}$,
x_{\alpha} \neq x_{\beta}$ for $\alpha \neq \beta$ and $x_{\alpha n} \prec x_{\alpha}$, where $\prec$ is some well-ordering on $C$ of the
same type as $\omega(c)$. Let $A = \{x_{\alpha} : \alpha < \omega(c)\}$.

For each $x$ in $C$, we construct a decreasing sequence $\{B_m(x) : m \in N\}$ of
countable subsets of $C$ containing $x$ as follows. For $x \not\in A$, we set $B_m(x) = \{x\}$. On $A$ we define $B_m$ by transfinite induction with respect to $\prec$ by setting for each
$\alpha < \omega(c)$,
$$B_m(x_{\alpha}) = \{x_{\alpha}\} \cup \bigcup_{n \geq 2m}(B_n(x_{\alpha n})) : m \in N\}.$$

It follows by transfinite induction that if $y \in B_m(x)$, then $B_n(y) \subseteq B_m(x)$ for
some $n$ in $N$, so that $\{B_m(x) : m \in N\}$ constitutes a local base of open sets at
$x$ with respect to some first countable, locally countable topology $\tau$ on $C$. It is
readily seen that $B_m(x)$ is $d$-closed and has $e$-diameter $\leq \frac{1}{m}$. Hence $\tau$ is finer than
the $e$-topology on $C$, $B_m(x)$ is $\tau$-clopen and $\text{ind}(C, \tau) = 0$. In fact, each infinite
sequence in $B_m(x)$ has an accumulation point in $B_m(x)$, so that $B_m(x)$ is $\tau$-compact
and $(C, \tau)$ is locally compact. In the sequel, $X$ denotes the space $(C, \tau)$.

Next, for each $i$ in $N$, we define a first countable topology $\tau_i$ finer than the
$d$-topology on $C$ by defining at each point $x$ a local base $\{B'_m(x) : m \in N\}$ consisting of
a decreasing sequence of $d$-closed sets containing $x$ as follows. If $x$ is in $\bigcup_{i=1}^\infty A_j$, we let $\{B'_m(x) : m \in N\}$ be a decreasing sequence of $d$-closed sets of $C$ forming a
local $d$-base at $x$. Otherwise, we let $B'_m(x) = B_m(x)$. Clearly, $\text{ind}(C, \tau_i) = 0$. In
the sequel, $X_i$ stands for the space $(C, \tau_i)$. It is readily verified that $(X_i, f_{ij})$, where
$f_{ij} : X_j \to X_i$ is the identity mapping, $i < j$, constitutes an inverse limit sequence
with limit space $X$.

The short proof of the following result is almost identical with the proof of Claim
1 of [1]. We give it here for completeness.

Claim 1. For each $i$ in $N$, $X_i$ is a Lindelöf space with $\dim X_i = 0$.

PROOF. Since every open set of $X_i$ containing a point of $A_{i+1}$ is a $d$-open
neighbourhood of that point, for any open cover $\mathcal{U}$ of $X_i$, we can choose $d$-open
sets $G_n, n$ in $N$, such that each $G_n$ is contained in some member of $\mathcal{U}$ and $A_{i+1} \subseteq
G = \bigcup_{n=1}^\infty G_n$. Since $X_i - G$ is an $e$-closed set of $C$ which does not intersect $A_{i+1}$,
it follows from Proposition 2 that $X_i - G$ is countable. This clearly implies that $X_i$ is Lindelöf and hence, since ind $X_i = 0$, dim $X_i = 0$.

Claim 2. $X$ is normal, countably paracompact and collectionwise normal.

Proof. Since $X - A_1$ is a clopen discrete subspace of $X$, it suffices to show that $A_1$ is normal, countably paracompact and collectionwise normal. Note that if $B_1, B_2, \ldots$ are closed subsets of $A_1$ with $\bigcap_{i=1}^{\infty} B_i = \emptyset$, then $\bigcap_{i=1}^{\infty} B_i$ is countable. For suppose that $\bigcap_{i=1}^{\infty} B_i$ is uncountable and hence has cardinality $c$. For each $i$ in $\mathbb{N}$, let $S_i$ be a countable $e$-dense subset of $B_i$. Then for some $\alpha < \omega(c)$, $(S_1, S_2, \ldots) = (S_{\alpha_1}, S_{\alpha_2}, \ldots)$ so that, by the definition of the topology $\tau$, $x_\alpha$ is an accumulation point of each $S_i$. Hence $x_\alpha \in \bigcap_{i=1}^{\infty} B_i$, a contradiction.

Let $E, F$ be disjoint closed sets of $A_1$. Then $A_1 \cap E^c \cap F^c$ is a countable zero set of $A_1$ and, using the fact that $A_1$ is locally countable, we can construct a countable cozero set $Z$ of $A_1$ containing it. Let $P$ be a cozero and $Q$ a zero set of $A_1$ with $A_1 \cap E^c \cap F^c \subset P \subset Q \subset Z$. Since ind $X = 0$, then ind $Z \leq 0$ and, since $Z$ is countable and therefore Lindelöf, we have dim $Z \leq 0$. Hence there exists a clopen set $Y$ of $Z$ such that $A_1 \cap E^c \cap F^c \subset Y \subset P$. Clearly, $Y$ is a closed subset of $Q$, and thus $Y$ is a countable clopen subset of $A_1$. Now $Y$ is Lindelöf and therefore normal, and hence there exist disjoint open subsets $G_1, H_1$ of $Y$ with $E \cap Y \subset G_1$ and $F \cap Y \subset H_1$. Also, there exist disjoint open subsets $G_2, H_2$ of $A_1 - Y$ with $A_1 \cap E^c - Y \subset G_2$ and $A_1 \cap F^c - Y \subset H_2$. Finally, $G = G_1 \cup G_2$ and $H = H_1 \cup H_2$ are disjoint open subsets of $A_1$ with $E \subset G$ and $F \subset H$. Thus $A_1$ is normal.

Let $\{B_i : i \in \mathbb{N}\}$ be a decreasing sequence of closed sets of $A_1$ with $\bigcap_{i=1}^{\infty} B_i = \emptyset$. $A_1$ is countably paracompact if there exists a decreasing sequence $\{W_i : i \in \mathbb{N}\}$ of open sets of $A_1$ with $B_i \subset W_i$ and $\bigcap_{i=1}^{\infty} W_i = \emptyset$ [3, Corollary 5.2.2]. Now $B = A_1 \cap \bigcap_{i=1}^{\infty} B_i$ is a countable subset of the locally countable space $A_1$ and hence it is contained in a countable open set $Y = \{y_1, y_2, \ldots\}$ of $A_1$, where $y_i \neq y_j$ for $i \neq j$. We may assume $B_i \cap B \subset \{y_i, y_{i+1}, \ldots\}$. Let $\{G_i : i \in \mathbb{N}\}$ be a decreasing sequence of $e$-open sets of $A_1 - B$ such that $A_1 \cap \bigcap_{i=1}^{\infty} G_i = \emptyset$, and put $W_i = G_i \cup \{y_i, y_{i+1}, \ldots\}$. Then $\{W_i : i \in \mathbb{N}\}$ is a decreasing sequence of open sets of $A_1$ with $B_i \subset W_i$ and $\bigcap_{i=1}^{\infty} W_i = \emptyset$. Hence $A_1$ is countably paracompact.

To prove that the normal space $A_1$ is collectionwise normal, it suffices to show that every discrete closed subset $B$ of $A_1$ is countable. Suppose the contrary, and let $S$ be a countable $e$-dense subset of $B$. Then $|S^c| = c$ and, for some $\alpha < \omega(c)$, $(S, S, \ldots) = (S_{\alpha_1}, S_{\alpha_2}, \ldots)$, so that, by the definition of $\tau$, $x_\alpha$ is an accumulation point of $B$, contradicting the fact that $B$ is discrete. This concludes the proof of Claim 2.

Claim 3. dim $X = \text{Ind} X = k$.

Proof. Let $E_i, F_i, i = 1, \ldots, k$, be the $e$-closed sets of $X$ occurring in Proposition 1. For each $i$, let $U_i, V_i$ be $e$-open sets of $X$ with $E_i \subset U_i, F_i \subset V_i$ and $\overline{U_i} \cap \overline{V_i} = \emptyset$, and suppose a closed subset $L_i$ of $A_1$ is a partition between $\overline{U_i}$ and $\overline{V_i}$. Write $L_i = M_i \cap N_i$ where $M_i, N_i$ are closed sets of $A_1$ with $\overline{U_i} \cap N_i = \emptyset, \overline{V_i} \cap M_i = \emptyset$ and $A_1 = M_i \cup N_i$. Then it follows from the fact that $A_1$ is $e$-dense in $X$ that $\overline{M_i} \cap \overline{N_i}$ is an $e$-partition between $E_i$ and $F_i$. Hence, by Proposition 1, $|\bigcap_{i=1}^{k} \overline{M_i} \cap \overline{N_i}| = c$, which implies that $\bigcap_{i=1}^{k} M_i \cap N_i = \bigcap_{i=1}^{k} L_i \neq \emptyset$. Now by the Eilenberg-Otton characterisation of covering dimension we have dim $X \geq k$. 
Consider next a closed subset $A$ of $A_1$ with $\dim(A, e) \leq n$. Let $E, F$ be disjoint closed subsets of $A$. Then $B = E^e \cap F^e \cap A$ is countable and hence is contained in a countable open set $Y$ of $A$. Clearly, $\dim Y \leq 0$ and hence, since also $A$ is normal, there is a clopen set $Z$ of $A$ with $B \subset Z \subset Y$. Also, there are disjoint open sets $P, Q$ of $Z$ such that $E \cap Z \subset P$, $F \cap Z \subset Q$ and $Z = P \cup Q$. Now $\text{Ind}(A - Z, e) \leq \text{Ind}(A, e) \leq n$, and so there are disjoint $e$-open sets $U, V$ of $A - Z$ such that $A \cap E^e - Z \subset U$, $A \cap F^e - Z \subset V$ and $\text{Ind}(A - (U \cup V \cup Z), e) \leq n - 1$. Then $G = U \cup P$, $H = V \cup Q$ are disjoint open sets of $A$ with $E \subset G$, $F \subset H$ and $\text{Ind}(A - (G \cup H), e) \leq n - 1$. It follows by induction that $\text{Ind} A \leq n$. In particular, $\text{Ind} A_1 \leq \text{Ind}(C, e) = k$ and hence, since the discrete space $X - A_1$ is clopen in $X$, $\text{Ind} X \leq k$. Hence, in view of the inequality $\dim X \leq \text{Ind} X$, which holds for all normal spaces, we have $\dim X = \text{Ind} X = k$.

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