PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 4, August 1982

THE DIMENSION OF INVERSE LIMIT AND N-COMPACT SPACES

M. G. CHARALAMBOUS

ABSTRACT. For each $k = 1, 2, \ldots, \infty$, we construct a normal N-compact space X with $\dim X = k$, where dim denotes covering dimension, which is the limit space of a sequence of zero-dimensional Lindelöf spaces.

Let X be the limit space of an inverse sequence (X_n, f_{nm}). In [1], we showed that $\dim X$ can be positive even if X is normal and X_n is Lindelöf and zero dimensional for each n in N, the set of natural numbers. In this paper we continue investigating the behaviour of covering dimension under inverse limits. We generalise the construction in [1] to obtain, for each $k = 1, 2, \ldots, \infty$, an inverse sequence (X_n, f_{nm}) of zero-dimensional Lindelöf spaces with limit space X such that X is normal and $\dim X = \operatorname{Ind} X = k$, where Ind denotes large inductive dimension. The space X is, in addition, first countable, locally compact, countably paracompact and collectionwise normal. Recall that a space is called N-compact if it is the inverse limit of countable discrete spaces. Every zero-dimensional Lindelöf space is N-compact, and so is the inverse limit of N-compact spaces. It follows that X is N-compact. N-compact spaces of positive covering dimension have previously been constructed in [5, 6] and also [7, 8], but our space X seems to be the first example showing that N-compact spaces can have infinite dimension.[1]

In this paper, all spaces are Tychonoff. The usual metric on the Cantor set C is denoted by d. I denotes the unit interval, $\omega(c)$ the first ordinal of cardinality c, the cardinality of the continuum, and $|X|$ the cardinality of a set X.

For standard results in Dimension Theory we refer to [4].

1. Preliminaries. The construction of the spaces X and X_n in this paper is only slightly more complicated than that of the corresponding spaces in [1]. In both papers, we draw from the techniques employed in [2, 7, 8]. The constructions in [1] made use of Wage's complete separable metric ρ on C which has the following properties: (a) the ρ-topology is finer than the usual topology on C, (b) every ρ-Borel set of C is d-Borel, and (c) every ρ-open set disjoint from a certain fixed ρ-closed set E has boundary of cardinality c. The constructions in the present paper are based on the existence of a separable metric e on C with the properties enumerated in the following result.

PROPOSITION 1. *For each $k = 1, 2, \ldots, \infty$, there exists a separable metric e on C with $d \leq e$ and k pairs of disjoint e-closed sets E_i, F_i, $i = 1, 2, \ldots, k$, such that*

Received by the editors September 21, 1981.

1980 *Mathematics Subject Classification*. Primary 54F45, 54G20.

Key words and phrases. Normal, Lindelöf, paracompact, N-compact space, covering and inductive dimension.

[1] The referee has informed the author that R. Engelking and E. Pol have recently constructed, for each $k \in \{1, 2, \ldots, \infty\}$ a Lindelöf, zero-dimensional space $X = X(k)$ such that $\dim X^2 = k$.

(1) *every uncountable e-closed and every nonempty e-open subset of C has cardinality c.*

(2) *Whenever L_i, $i = 1, \dots, k$, is an e-partition between E_i and F_i, then*

$$\left|\bigcap_{i=1}^{k} L_i\right| = c.$$

(3) $\dim(C, e) = k$.

N.B. The condition $d \leq e$ implies that the e-topology is finer than the usual topology on C.

PROOF. Let $Y = C \times I^k$ and $\pi = Y \to C$ the canonical projection. Let $\mathcal{G}$ be the collection of all subsets G of Y such that $|\pi(G)| = c$ and G is either open or closed. Note that $|\mathcal{G}| = c$ and we may choose an enumeration $\{G_\alpha \colon \alpha < \omega(c)\}$ of $\mathcal{G}$ such that for each G in $\mathcal{G}$, $G = G_\alpha$ for c ordinals $\alpha < \omega(c)$. For each $\alpha < \omega(c)$, since $|\pi(G_\alpha)| = c$, we can choose by transfinite induction, a point x_α in $\pi(G_\alpha)$ so that $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$. Next, we define a function $f \colon C \to I^k$ as follows. If $x = x_\alpha$ for some $\alpha < \omega(c)$, we choose $f(x)$ so that $(x, f(x))$ is in G_α. If not, we set $f(x) = 0$.

Let X be the subspace $\{(x, f(x)) \colon x \in C\}$ of Y. Then $\pi \colon X \to C$ is bijective and continuous. Consider an uncountable closed set E of X. Then $E = F \cap X$ for some closed subset F of Y. Now the closed set $\pi(F)$ of C is uncountable and hence has cardinality c, so that F is in $\mathcal{G}$. If $A = \{\alpha < \omega(c) \colon F = G_\alpha\}$, then E contains $\{(x_\alpha, f(x_\alpha) \colon \alpha \in A\}$, and since $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, and $|A| = c$, we have $|E| = c$. Similarly, every nonempty open subset of X has cardinality c. In fact, $|G \cap X| = c$ for every G in $\mathcal{G}$.

Let A_i, B_i, $i = 1, \dots, k$, be the pairs of opposite faces of I^k. Let U_i, V_i be open sets of Y such that $C \times A_i \subset U_i$, $C \times B_i \subset V_i$ and $\overline{U}_i \cap \overline{V}_i = \emptyset$. Set $E_i = X \cap \overline{U}_i$, $F_i = X \cap \overline{V}_i$, $i = 1, \dots, k$, and suppose L_i is a partition in X between E_i and F_i. Then there exist disjoint open sets G_i, H_i of Y with $E_i \subset G_i$, $F_i \subset H_i$ and $X - (G_i \cup H_i) \subset L_i$. Let $P_i = U_i \cup (G_i - \overline{V}_i)$, $Q_i = V_i \cup (H_i - \overline{U}_i)$ and $M_i = Y - (P_i \cup Q_i)$. Then M_i is a partition in Y between $C \times A_i$ and $C \times B_i$ with $M_i \cap X \subset L_i$. It follows that $\bigcap_{i=1}^k M_i$ contains at least one point from $\{x\} \times I^k$ for each x in C. Thus $|\pi(\bigcap_{i=1}^k M_i)| = c$, which implies that

$$\left|\bigcap_{i=1}^{k} M_i \cap X\right| = \left|\bigcap_{i=1}^{k} L_i\right| = c.$$

Now, by the Otto-Eilenberg characterisation of covering dimension [4, Theorem 1.7.9], $\dim X \geq k$ and, since X is a subspace of the k-dimensional space Y, we have $\dim X = k$.

Finally, to complete the proof of Proposition 1, it suffices to let, for x, y in C, $e(x, y)$ denote the usual Euclidean distance between the points $(x, f(x))$ and $(y, f(y))$.

We shall need the following result, which is the analogue for the metric e on C of the well-known theorem on the existence of Bernstein sets in a complete separable metric space.

PROPOSITION 2. *There is a partition $A_1, A_2, A_3, \dots$ of C such that $|A_i \cap F| = c$ for each i in N and each uncountable e-closed set F of C.*

PROOF. Let $\mathcal{F}$ be the collection of all uncountable e-closed sets of C. Let $\{F_\alpha : \alpha < \omega(c)\}$ be an enumeration of $\mathcal{F}$ such that for each F in $\mathcal{F}$, $F = F_\alpha$ for c ordinals $\alpha < \omega(c)$. For each α, by Proposition 1, $|F_\alpha| = c$ and thus we can choose by transfinite induction points $x_{\alpha 1}, x_{\alpha 2}, \dots$ in F_α so that $x_{\alpha n} \neq x_{\beta m}$ if $(\alpha, n) \neq (\beta, m)$. It now suffices to let $A_i = \{x_{\alpha i} : \alpha < \omega(c)\}$ for $i = 2, 3, \dots,$ and $A_1 = C - \bigcup_{i=2}^{\infty} A_i$.

N.B. It follows from Proposition 1(1), that each A_i is e-dense in C.

2. The construction. In the sequel, k denotes a fixed number of the set $\{1, 2, \dots, \infty\}$, e the metric on C given by Proposition 1, and $A_1, A_2, \dots$ the partition of C provided by Proposition 2.

Let $\{(S_{\alpha 1}, S_{\alpha 2}, \dots) : \alpha < \omega(c)\}$ be the collection of all sequences of countable subsets of A_1 with $|\bigcap_{i=1}^{\infty} \overline{S}_{\alpha i}^e| = c$. Since

$$\left| A_1 \cap \bigcap_{i=1}^{\infty} \overline{S}_{\alpha i}^e \right| = c,$$

for each $\alpha < \omega(c)$, we can choose x_α in $A_1 \cap \bigcap_{i=1}^{\infty} \overline{S}_{\alpha i}^e$ and a sequence $\{x_{\alpha n}\}$ in A_1 so that $e(x_\alpha, x_{\alpha n}) < \frac{1}{n}$, $\{x_{\alpha n}\}$ contains infinitely many points from each $S_{\alpha i}$, $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$ and $x_{\alpha n} \lhd x_\alpha$, where $\lhd$ is some well-ordering on C of the same type as $\omega(c)$. Let $A = \{x_\alpha : \alpha < \omega(c)\}$.

For each x in C, we construct a decreasing sequence $\{B_m(x) : m \in N\}$ of countable subsets of C containing x as follows. For $x \notin A$, we set $B_m(x) = \{x\}$. On A we define B_m by transfinite induction with respect to $\lhd$ by setting for each $\alpha < \omega(c)$,

$$B_m(x_\alpha) = \{x_\alpha\} \cup \bigcup (B_n(x_{\alpha n}) : n \geq 2m).$$

It follows by transfinite induction that if $y \in B_m(x)$, then $B_n(y) \subset B_m(x)$ for some n in N, so that $\{B_m(x) : m \in N\}$ constitutes a local base of open sets at x with respect to some first countable, locally countable topology τ on C. It is readily seen that $B_m(x)$ is d-closed and has e-diameter $\leq \frac{1}{m}$. Hence τ is finer than the e-topology on C, $B_m(x)$ is τ-clopen and $\operatorname{ind}(C, \tau) = 0$. In fact, each infinite sequence in $B_m(x)$ has an accumulation point in $B_m(x)$, so that $B_m(x)$ is τ-compact and (C, τ) is locally compact. In the sequel, X denotes the space (C, τ).

Next, for each i in N, we define a first countable topology τ_i finer than the d-topology on C by defining at each point x a local base $\{B_m^i(x) : m \in N\}$ consisting of a decreasing sequence of d-closed sets containing x as follows. If x is in $\bigcup_{j=i+1}^{\infty} A_j$, we let $\{B_m^i(x) : m \in N\}$ be a decreasing sequence of d-clopen sets of C forming a local d-base at x. Otherwise, we let $B_m^i(x) = B_m(x)$. Clearly, $\operatorname{ind}(C, \tau_i) = 0$. In the sequel, X_i stands for the space (C, τ_i). It is readily verified that (X_i, f_{ij}), where $f_{ij} : X_j \to X_i$ is the identity mapping, $i < j$, constitutes an inverse limit sequence with limit space X.

The short proof of the following result is almost identical with the proof of Claim 1 of [**1**]. We give it here for completeness.

Claim 1. For each i in N, X_i is a Lindelöf space with $\dim X_i = 0$.

PROOF. Since every open set of X_i containing a point of A_{i+1} is a d-open neighbourhood of that point, for any open cover $\mathcal{U}$ of X_i, we can choose d-open sets G_n, n in N, such that each G_n is contained in some member of $\mathcal{U}$ and $A_{i+1} \subset G = \bigcup_{n=1}^{\infty} G_n$. Since $X_i - G$ is an e-closed set of C which does not intersect A_{i+1}, it follows from Proposition 2 that $X_i - G$ is countable. This clearly implies that X_i is Lindelöf and hence, since $\operatorname{ind} X_i = 0$, $\dim X_i = 0$.

Claim 2. X is normal, countably paracompact and collectionwise normal.

PROOF. Since $X - A_1$ is a clopen discrete subspace of X, it suffices to show that A_1 is normal, countably paracompact and collectionwise normal. Note that if $B_1, B_2, \dots$ are closed subsets of A_1 with $\bigcap_{i=1}^{\infty} B_i = \emptyset$, then $\bigcap_{i=1}^{\infty} \overline{B}_i^e$ is countable. For suppose that $\bigcap_{i=1}^{\infty} \overline{B}_i^e$ is uncountable and hence has cardinality c. For each i in N, let S_i be a countable e-dense subset of B_i. Then for some $\alpha < \omega(c)$, $(S_1, S_2, \dots) = (S_{\alpha 1}, S_{\alpha 2}, \dots)$ so that, by the definition of the topology τ, x_α is an accumulation point of each S_i. Hence $x_\alpha \in \bigcap_{i=1}^{\infty} B_i$, a contradiction.

Let E, F be disjoint closed sets of A_1. Then $A_1 \cap \overline{E}^e \cap \overline{F}^e$ is a countable zero set of A_1 and, using the fact that A_1 is locally countable, we can construct a countable cozero set Z of A_1 containing it. Let P be a cozero and Q a zero set of A_1 with $A_1 \cap \overline{E}^e \cap \overline{F}^e \subset P \subset Q \subset Z$. Since $\operatorname{ind} X = 0$, then $\operatorname{ind} Z \leq 0$ and, since Z is countable and therefore Lindelöf, we have $\dim Z \leq 0$. Hence there exists a clopen set Y of Z such that $A_1 \cap \overline{E}^e \cap \overline{F}^e \subset Y \subset P$. Clearly, Y is a closed subset of Q, and thus Y is a countable clopen subset of A_1. Now Y is Lindelöf and therefore normal, and hence there exist disjoint open subsets G_1, H_1 of Y with $E \cap Y \subset G_1$ and $F \cap Y \subset H_1$. Also, there exist disjoint e-open subsets G_2, H_2 of $A_1 - Y$ with $A_1 \cap \overline{E}^e - Y \subset G_2$ and $A_1 \cap \overline{F}^e - Y \subset H_2$. Finally, $G = G_1 \cup G_2$ and $H = H_1 \cup H_2$ are disjoint open subsets of A_1 with $E \subset G$ and $F \subset H$. Thus A_1 is normal.

Let $\{B_i : i \in N\}$ be a decreasing sequence of closed sets of A_1 with $\bigcap_{i=1}^{\infty} B_i = \emptyset$. A_1 is countably paracompact if there exists a decreasing sequence $\{W_i : i \in N\}$ of open sets of A_1 with $B_i \subset W_i$ and $\bigcap_{i=1}^{\infty} W_i = \emptyset$ [3, Corollary 5.2.2]. Now $B = A_1 \cap \bigcap_{i=1}^{\infty} \overline{B}_i^e$ is a countable subset of the locally countable space A_1 and hence it is contained in a countable open set $Y = \{y_1, y_2, \dots\}$ of A_1, where $y_i \neq y_j$ for $i \neq j$. We may assume $B_i \cap B \subset \{y_i, y_{i+1}, \dots\}$. Let $\{G_i : i \in N\}$ be a decreasing sequence of e-open sets of $A_1 - B$ such that $A_1 \cap \overline{B}_i^e - B \subset G_i$ and $\bigcap_{i=1}^{\infty} G_i = \emptyset$, and put $W_i = G_i \cup \{y_i, y_{i+1}, \dots\}$. Then $\{W_i : i \in N\}$ is a decreasing sequence of open sets of A_1 with $B_i \subset W_i$ and $\bigcap_{i=1}^{\infty} W_i = \emptyset$. Hence A_1 is countably paracompact.

To prove that the normal space A_1 is collectionwise normal, it suffices to show that every discrete closed subset B of A_1 is countable. Suppose the contrary, and let S be a countable e-dense subset of B. Then $|\overline{S}^e| = c$ and, for some $\alpha < \omega(c)$, $(S, S, \dots) = (S_{\alpha 1}, S_{\alpha 2}, \dots)$, so that, by the definition of τ, x_α is an accumulation point of B, contradicting the fact that B is discrete. This concludes the proof of Claim 2.

Claim 3. $\dim X = \operatorname{Ind} X = k$.

PROOF. Let E_i, F_i, $i = 1, \dots, k$, be the e-closed sets of X occurring in Proposition 1. For each i, let U_i, V_i be e-open sets of X with $E_i \subset U_i$, $F_i \subset V_i$ and $\overline{U}_i^e \cap \overline{V}_i^e = \emptyset$, and suppose a closed subset L_i of A_1 is a partition between $\overline{U}_i^e$ and $\overline{V}_i^e$. Write $L_i = M_i \cap N_i$ where M_i, N_i are closed sets of A_1 with $\overline{U}_i^e \cap N_i = \emptyset$, $\overline{V}_i^e \cap M_i = \emptyset$ and $A_1 = M_i \cup N_i$. Then it follows from the fact that A_1 is e-dense in X that $\overline{M}_i^e \cap \overline{N}_i^e$ is an e-partition between E_i and F_i. Hence, by Proposition 1, $|\bigcap_{i=1}^{k} \overline{M}_i^e \cap \overline{N}_i^e| = c$, which implies that $\bigcap_{i=1}^{k} M_i \cap N_i = \bigcap_{i=1}^{k} L_i \neq \emptyset$. Now by the Eilenberg-Otton characterisation of covering dimension we have $\dim X \geq k$.

Consider next a closed subset A of A_1 with $\dim(A, e) \leq n$. Let E, F be disjoint closed subsets of A. Then $B = \overline{E}^e \cap \overline{F}^e \cap A$ is countable and hence is contained in a countable open set Y of A. Clearly, $\dim Y \leq 0$ and hence, since also A is normal, there is a clopen set Z of A with $B \subset Z \subset Y$. Also, there are disjoint open sets P, Q of Z such that $E \cap Z \subset P$, $F \cap Z \subset Q$ and $Z = P \cup Q$. Now $\operatorname{Ind}(A - Z, e) \leq \operatorname{Ind}(A, e) \leq n$, and so there are disjoint e-open sets U, V of $A - Z$ such that $A \cap \overline{E}^e - Z \subset U$, $A \cap \overline{F}^e - Z \subset V$ and $\operatorname{Ind}(A - (U \cup V \cup Z), e) \leq n - 1$. Then $G = U \cup P$, $H = V \cup Q$ are disjoint open sets of A with $E \subset G$, $F \subset H$ and $\operatorname{Ind}(A - (G \cup H), e) \leq n - 1$. It follows by induction that $\operatorname{Ind} A \leq n$. In particular, $\operatorname{Ind} A_1 \leq \operatorname{Ind}(C, e) = k$ and hence, since the discrete space $X - A_1$ is clopen in X, $\operatorname{Ind} X \leq k$. Hence, in view of the inequality $\dim \leq \operatorname{Ind}$, which holds for all normal spaces, we have $\dim X = \operatorname{Ind} X = k$.

REFERENCES

1. M. G. Charalambous, *An example concerning inverse limit sequences of normal spaces*, Proc. Amer. Math. Soc. **78** (1980), 605–607.
2. E. van Douwen, *A technique for constructing honest locally compact submetrizable examples*, preprint.
3. R. Engelking, *General topology*, PWN, Warsaw, 1977.
4. ——, *Dimension theory*, North-Holland, Amsterdam, 1978.
5. S. Mrowka, *Recent results on E-compact spaces*, General Topology and its Applications (Proc. Second Pittsburgh Internat. Conf.), Lecture Notes in Math., vol. 378, Springer-Verlag, Berlin and New York, 1974, pp. 298–301.
6. E. Pol and R. Pol, *A hereditarily normal strongly zero-dimensional space with a subspace of positive dimension and an N-compact space of positive dimension*, Fund. Math. **97** (1977), 43–50.
7. T. Przymusiński, *On the dimension of product spaces and an example of M. Wage*, Proc. Amer. Math. Soc. **76** (1979), 315–321.
8. M. Wage, *The dimension of product spaces*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 4671–4672.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NAIROBI, NAIROBI, P.O. BOX 30197, KENYA