SIGMA-COMPACT LOCALLY CONVEX METRIC LINEAR SPACES
UNIVERSAL FOR COMPACTA ARE HOMEOMORPHIC

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ABSTRACT. It is proved that every σ-compact locally convex metric linear space containing a topological copy of the Hilbert cube Q is homeomorphic to Σ the linear span of the Hilbert cube Q, in the Hilbert space l_2.

Let us fix the following representation of the Hilbert cube Q in the Hilbert space l_2:

$$Q = \{(x_n) \in l_2 \colon |x_n| \leq n^{-2} \text{ for every } n\}.$$

Consider the space $\Sigma = \mathrm{span}(Q) \subset l_2$. We note that Σ is a σ-compact linear dense subspace of l_2. The purpose of this note is to prove the following fact:

THEOREM. *Every σ-compact locally convex metric linear space E containing a topological copy of the Hilbert cube Q is homeomorphic to Σ. Moreover, if $\tilde{E}$ is the completion of E, then the pairs $(\tilde{E}, E)$ and (l_2, Σ) are homeomorphic.*

The Theorem resolves the outstanding problem LS3 in [4] posed by Anderson in connection with the fact that the assertion of the Theorem was known to be true under the additional assumption that the copy of the Hilbert cube above was a convex set (see: Anderson [1], Bessaga-Pełczyński [2] and Toruńczyk [5]). Let us recall that the classical theorem of Keller-Klee [3, p. 100] states that every metrizable infinite-dimensional compact convex subset K of a locally convex space is homeomorphic to Q. We complete the paper with an example of a σ-compact dense linear subspace of l_2 which contains a topological copy of Q and does not contain any infinite-dimensional compact convex subset. This shows that the Theorem cannot be reduced to the above-mentioned result of [1, 2, 5].

The proof of the Theorem involves the technique of absorbing sets. We will use the approach of West [8] and Toruńczyk [6] instead of the original approaches of Anderson (cap-set technique [1]) and Bessaga-Pełczyński (skeleton technique [2]). According to [8, 6] a σ-compact set M is an absorbing set (for compacta) in a space X homeomorphic to l_2 if the following condition is satisfied:

(abs) *every map $f\colon Q \to X$ such that $f|B$ is an embedding of a closed set $B \subset Q$ into $M \subset X$ can be approximated by embeddings $g\colon Q \to M$ with $g|B = f|B$.*

The space Σ may serve as an example of an absorbing set in l_2 (see [3, p. 275]). We intend to show that the space E is also an absorbing set in the completion $\tilde{E}$ homeomorphic, by the Kadec-Anderson theorem [3, p. 189], to l_2. Then the assertion of our Theorem will follow from the theorem on equivalence of absorbing sets [8, 6] stating that: if M_i is an absorbing set in X_i ($i = 1, 2$), then the pairs (X_1, M_1) and (X_2, M_2) are homeomorphic.

The verification of the condition (abs) makes use of the Proposition below. We shall assume that $|\cdot|$ is an F-norm on a locally convex space E, i.e., a function $|\cdot|: E \to [0, \infty)$ such that $(x, y) \to |x - y|$ defines a compatible metric on E. We shall additionally require that each ball

$$N(\eta) = \{x \in E: |x| < \eta\}$$

is convex. We shall often write, for φ, $\psi: Y \to E$

$$|\varphi - \psi| = \sup\{|\varphi(y) - \psi(y)|: y \in Y\}.$$

The symbol $H(Y)$ denotes the space of all self-homeomorphisms of Y.

PROPOSITION. *Let $(E, |\cdot|)$ be an infinite-dimensional locally convex space and let B and A be disjoint compacta of E. Then, given a map $f: B \to E$ and $\epsilon > 0$, there exists a homeomorphism $h \in H(E)$ satisfying*
  (i) $|h - \mathrm{id}| < |f - \mathrm{id}_B| + \epsilon$;
  (ii) $h|A = \mathrm{id}_A$;
  (iii) $|h|B - f| < \epsilon$.

LEMMA 1. *Assuming that $\mathrm{span}(B \cup f(B))$ is finite-dimensional and that f is an embedding with $f(B) \cap A = \emptyset$, there exists $h \in H(E)$ extending f and satisfying the conditions* (i) *and* (ii) *of the Proposition.*

LEMMA 2. *Given a positive δ, there exists a map*

$$u: E \times [0, 1) \cup B \times \{1\} \to E$$

such that, writing $u_t = u(\cdot, t)$, we have $u_1(B) \cap A = \emptyset$, $\mathrm{span}(u_1(B))$ is finite-dimensional and, for $t < 1$,
  (iv) $u_t \in H(E)$ *with $u_0 = \mathrm{id}$;*
  (v) $|u_t - \mathrm{id}| < \delta$;
  (vi) $u_t|A = \mathrm{id}_A$.

PROOF OF LEMMA 1. Pick a positive η with $\eta < \mathrm{dist}(A, B \cup f(B))$ and $\eta < \epsilon$. The compactness of A implies that there is a finite-dimensional linear subspace E' of E satisfying

(1) $A \subset E' + N(\eta/16)$

and such that $E_1 = \mathrm{span}(B \cup f(B)) \subset E'$ and $\dim(E') \geq 2 \cdot \dim(E_1) + 3$. Consider the set $N(\eta/2) \backslash (E' + N(\eta/4)) = C$. If $C \neq \emptyset$, pick a point $x_0 \notin E' + N(\eta/4)$ to have

(2) $|x_0| < \eta/2.$

Otherwise, there exists a point $x_0 \in E \backslash E'$ such that

(2') $|sx_0| < \eta/2 \quad$ for every $s \geq 0.$

In either case, let $E_0 = \mathrm{span}(E', \{x_0\})$. By a theorem of Michael [3, p. 85], there exists a continuous right inverse $\varphi: E/E_0 \to E$ for the quotient map $\kappa: E \to E/E_0$; by the local convexity of E we may additionally require that

(3) $|\varphi(\kappa(x))| \leq 2|x| \quad$ for all $x \in E.$

Writing $r(x) = x - \varphi(\kappa(x))$, by (1) and (3), we have

(4) $$r(A) \subset A + N(\eta/8).$$

The formula

$$\alpha_t(b) = \begin{cases} b + tx_0, & t \in [0, 1], \\ (2 - t) \cdot (b + x_0) + (t - 1) \cdot (f(b) + x_0), & t \in [1, 2], \\ f(b) + (3 - t)x_0, & t \in [2, 3], \end{cases}$$

defines a homotopy α joining id_B with f in E_0. By (2), we have

(5) $$|\alpha_t(b) - b| < \eta/2 + |f(b) - b| + \eta/2 < \epsilon + |f(b) - b|$$

for every $b \in B$. Moreover, by (4), (2) or (2') we get

(6) $$\alpha(b \times [0, 3]) \cap r(A) = \emptyset.$$

(In the case where $C = \emptyset$, in the formula describing α, we put instead of x_0, the point sx_0 in order to have $(sx_0 + E') \cap r(A) = \emptyset$.) The space E_0 has the estimated extension homeomorphism property for compacta K with $\dim(\mathrm{span}(K)) \leq \dim(E_1) + 1$ (we apply, for instance, the "graph-trick" of Klee [3, p. 62]; see also [4, p. 11]. Consequently, there exists a homeomorphism $\bar{h} \in H(E_0)$ with $\bar{h}|B = f$ satisfying, by (5),

$$|\bar{h} - \mathrm{id}| < \epsilon + |f - \mathrm{id}_B|$$

and such that, by (6), $\bar{h}|r(A) = \mathrm{id}_{r(A)}$. Finally, we put $h(x) = \varphi(\kappa(x)) + \bar{h}(r(x))$ for $x \in E$.

PROOF OF LEMMA 2. Assume that $\delta < \mathrm{dist}(A, B)$ and consider a finite-dimensional linear space $E_0 \subset E$ with

(7) $$A \cup B \subset E_0 + N(\delta/4).$$

Let $\varphi: E/E_0 \to E$ be a map of the proof of Lemma 1 (i.e. φ satisfies the condition (3)). Write $h_0(x) = (\kappa(x), r(x))$ for the homeomorphism of E onto $E/E_0 \times E_0$ with $r(x) = x - \varphi(\kappa(x))$. Let $|\cdot|$ be the quotient F-norm on $E/E_0 = Y$. By (7), we have $r(A) \cap r(B) = \emptyset$. Let $\omega: E_0 \to [0, \delta/4]$ be a map with

$$h_0(B) \subset \{(y, e) \in Y \times E_0 : |y| < \omega(e)\} \subset h_0(E \backslash A).$$

Let $\{0 < \alpha_t \leq 1\}$, $0 \leq t < 1$, be a homotopy of $[0, \infty]$ with $\alpha_0 \equiv 1$, α_t is monotone and $\alpha_t|[1, \infty] = 1$ for each t and such that for each $s < 1$, $\lim_{t \to 1} \alpha_t(s) = 0$. Define an isotopy $\{g_t\}$, $0 \leq t < 1$, of $Y \times E_0$ by the formula

$$g_t(y, e) = \begin{cases} (\alpha_t(|y|/\omega(e)) \cdot y, e), & \text{if } y \neq 0, \\ (0, e), & \text{if } y = 0. \end{cases}$$

Then the desired map u may be defined by $u(x, t) = h_0^{-1} g_t h_0(x)$ for $0 \leq t < 1$, and $u(b, 1) = r(b)$ for $b \in B$.

PROOF OF PROPOSITION. (1) Assume $\mathrm{span}(B \cup f(B)) = E_1$ is finite-dimensional. One can easily approximate f by maps with ranges disjoint from A. Thus, we will require that $f(B) \cap A = \emptyset$. Since B is finite-dimensional, there exists an embedding $w: B \to E$ with $|w(b)| < \delta$ and such that $E_2 = \mathrm{span}(w(B))$ is finite-dimensional with $E_2 \cap E_1 = \{0\}$. Writing $v = f + w$, we see that v is an

embedding satisfying

(8) $|v - \mathrm{id}_B| < |f - \mathrm{id}| + \delta$ and $|v - f| < \delta.$

If δ is sufficiently small, we also get

(9) $v(B) \cap A = \emptyset.$

Assuming $\delta < \epsilon$ and applying Lemma 1, we extend the embedding v to a required homeomorphism h.

(2) The general case. Let $\bar{f}\colon E \to E$ be a map with $\bar{f}|B = f$ (we use the AR-property of E). Pick δ with $0 < \delta < \epsilon/8$ and such that

(10) $b \in B,$ $|x - b| < \delta$ implies $|\bar{f}(x) - f(b)| < \epsilon/4.$

With this δ, consider a homotopy u_t of Lemma 2 and write $B' = u_1(B)$. Let us approximate $\bar{f}|B'$ by $f'\colon B' \to E$ such that

(11) $|\bar{f}|B' - f'| < \epsilon/4$ and $\dim(\mathrm{span}(f'(B'))) < \infty.$

Now, observing that the triple (B', A, f') satisfies the assumptions of the previous case, we conclude that there exists $h' \in H(E)$ satisfying $|h'|B' - f'| < \epsilon/4$, $|h' - \mathrm{id}| < |f' - \mathrm{id}_{B'}| + \epsilon/4$ and $h'|A = \mathrm{id}_A$. We claim that $h_t = h' \circ u_t$ may serve as a required homeomorphism, provided that $t \neq 1$ is sufficiently close to 1.

First observe that for $b' = u_1(b) \in B'$, we can estimate, using (11), (v) and (10),

$$|f'(b') - b'| \leq |f'(b') - \bar{f}(b')| + |\bar{f}(u_1(b)) - f(b)| + |f(b) - b|$$
$$+ |b - u_1(b)| \leq \epsilon/4 + \epsilon/4 + |f - \mathrm{id}_B| + \epsilon/8.$$

Consequently, by (v), for all $t < 1$, we have

$$|h_t - \mathrm{id}| \leq |h' \circ u_t - u_t| + |u_t - \mathrm{id}| \leq |h' - \mathrm{id}| + \epsilon/8$$
$$\leq 5\epsilon/8 + |f - \mathrm{id}_B| + \epsilon/4 + \epsilon/8 \leq |f - \mathrm{id}_B| + \epsilon.$$

Now, we take $t_0 \in (0, 1)$ such that for every $t \geq t_0$ we have

(12) $|h'(u_t(b)) - h'(u_1(b))| < \epsilon/4$ for all $b \in B.$

Using (12)–(10), for $t \in [t_0, 1)$, we have

$$|h_t(b) - f(b)| = |h'u_t(b) - f(b)| \leq |h'u_t(b) - h'u_1(b)|$$
$$+ |h'u_1(b) - f'u_1(b)| + |f'(u_1(b)) - \bar{f}(u_1(b))|$$
$$+ |\bar{f}u_1(b) - f(b)| < \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4.$$

Finally, since both h' and u_t are the identity on A, the composition $h_t = h' \circ u_t$ has this property also.

PROOF OF THEOREM. By what was said in the Introduction we have to verify the condition (abs) for the pair $(X, M) = (\tilde{E}, E)$. Since the set $\tilde{E} \backslash E$ is locally homotopy negligible in $\tilde{E}$ (see [7, Remark 2.9 and Theorem 2.4]), the map $f\colon Q \to \tilde{E}$ can be approximated by maps which are equal to f on B and have ranges in E. Using our hypothesis, we may assume that the Hilbert cube is placed in the space E; possibly with $Q \cap f(Q) = \emptyset.$

Fix a positive ϵ. Consider a tower $\{A_i\}_{i=0}^\infty$ consisting of compacta, with $A_0 = \emptyset$ and $\bigcup_{i=0}^\infty A_i = Q \backslash B$. We shall inductively construct a sequence of homeomorphisms $\{h_n\}_{n=0}^\infty \subset H(E)$ such that, writing $g_n = h_n \circ h_{n-1} \circ \cdots \circ h_0$, the following

conditions are satisfied:

(13) $$|g_n - g_{n-1}| < 2^{-n+1}\epsilon$$

and

(14) $$g_n|A_{n-1} = g_{n-1}|A_{n-1} \quad \text{for } n = 1, 2, \ldots;$$

(15) $$|g_n|B - f|B| < 2^{-n-1}\epsilon$$

and

(16) $$g_n|f(Q) = \text{id} \quad \text{for } n = 0, 1, 2, \ldots.$$

Let h_0 be a homeomorphism obtained by the Proposition applied to the quadruple $(Q, f, f(Q), \epsilon/2)$. Assume $h_0, \ldots, h_{n-1}$ $(n \geq 1)$ are already constructed. To get h_n, apply the Proposition to the quadruple $(g_{n-1}(B), f \circ g_{n-1}^{-1}|g_{n-1}(B), g_{n-1}(A_n) \cup f(Q), 2^{-n-1}\epsilon)$. Then (13) is a consequence of the calculation

$$|g_n - g_{n-1}| = |h_n - \text{id}| < |f \circ g_{n-1}^{-1}|g_{n-1}(B) - \text{id}_{g_{n-1}(B)}| + 2^{-n-1}\epsilon$$
$$\leq |f|B - g_{n-1}|B| + 2^{-n-1}\epsilon \leq 2^{-n}\epsilon + 2^{-n-1}\epsilon < 2^{-n+1}\epsilon.$$

The conditions (14)–(16) also follow.

Now, put $g(q) = \lim g_n(q)$ for $q \in Q$. By (13)–(15), g is a continuous map from Q into E such that $g|B = f|B$ and $|g - f| < 3\epsilon$. Finally, by (14) and (16), g is a 1-1 map (and therefore an embedding of Q into E).

EXAMPLE. Let $i: Q \to l_2$ be an embedding of the Hilbert cube Q into the Hilbert space l_2 such that the set $Q' = i(Q)$ is linearly independent (see [3, p. 193] and the Note below). Let $E = \text{span}(Q') \subset l_2$. Since E can be expressed $E = \bigcup_{n=1}^{\infty} A_n$, with $A_n = \{\sum_{k=1}^{n} t_k \cdot q'_k : |t_k| \leq n, q'_k \in Q'\}$, E is σ-compact. The space E does not contain any infinite-dimensional compact convex subset.

PROOF. First, we verify that every convex subset C contained in A_n has dimension $\leq n$. Let m be the maximal integer such that there is $c = \sum_{k=1}^{m} t_k \cdot q'_k \in C$ with $t_k \neq 0$. We claim that $C \subset \text{span}(\{q'_1, \ldots, q'_m\})$. If there were $c_0 \in C$ such that $c_0 = \sum_{k=1}^{m} s_k \cdot q'_k + \sum_{k=m+1}^{l} s_k \cdot q'_k$ with $s_k \neq 0$ for $k = m+1, \ldots, l$, then we would find a suitable $t \in (0, 1)$ such that the point $tc + (1 - t)c_0 \in C$ would have all l coordinates different from 0. This contradicts the maximality of m.

Now, assuming that E contains an infinite-dimensional compact convex subset, by a Baire category argument, there is A_{n_0} which also contains such a subset. However, by the above argument, this is impossible.

Note. Let E be a locally convex metric space containing an infinite-dimensional compact convex set C. Then, E contains a dense σ-compact linear subspace E_0 such that: (a) E_0 contains a topological copy of the Hilbert cube; and (b) E_0 does not contain any infinite-dimensional compact convex subset. This is a consequence of the fact that there exists an embedding of Q onto a linearly independent subset C' of C with $\text{cl}(\text{span}(C')) \supset \text{span}(C)$. (To see this, consider C as a subset of l_2 with $0 \in C$ (use [3, p. 100]) and take an orthogonal set $\{e_i\}_{i=1}^{\infty} \subset C$ which is total for $\text{span}(C)$. The formula $(x_n) \to \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x_n^k \cdot e_{n(k)}$, where $\{n(k): k = 1, 2, \ldots\}$ is a partition of the integers into countable sets, defines a required embedding.

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