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THE SPACES WHICH CONTAIN AN S-SPACE

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ABSTRACT. Under the continuum hypothesis, we show that a T_1-space X contains an S-space if and only if there is an uncountable locally countable set $E \subset X$ containing no Borel subset of X.

All spaces in this note are T_1. A space X is called *locally countable* if each point has a countable neighborhood. A set $E \subset X$ is called locally countable if it is such as a subspace. An S-*space* is a hereditarily separable space which is not Lindelöf. We remark that contrary to the usual definition, in this note an S-space is *not* required to be regular. If X is a space, we denote by $\mathcal{B}(X)$ the family of all *Borel subsets* of X, i.e., the σ-algebra in X generated by the topology of X. If A is a set, we denote by $|A|$ its cardinality.

The *Zermelo-Fraenkel set theory* including the *axiom of choice*, the *continuum hypothesis*, and *Martin's axiom* will be abbreviated as ZFC, CH, and MA, respectively.

1. DEFINITION. A space X is called *ample* if each uncountable locally countable set $E \subset X$ contains an uncountable subset $B \in \mathcal{B}(X)$.

In this definition the word "ample" is used to indicate a certain richness of the Borel structure of X. The ample spaces are important in the topological measure theory; for each diffused, regular, Radon measure in an ample space is σ-finite (see [**GP**, 2.4]). Thus it appears useful to characterize the ample spaces in terms of much studied S-spaces.

The following lemma was first proved by R. J. Gardner (see [**GP**, 2.5]).

2. LEMMA. *Let $E \subset X$ be a locally countable set containing no uncountable subset $B \in \mathcal{B}(X)$. Then E is hereditarily separable.*

PROOF. It suffices to show that E is separable. By Zorn's lemma there is a maximal disjoint family $\mathcal{D}$ of nonempty countable subsets of E which are open in E. By the maximality of $\mathcal{D}$, $\bigcup \mathcal{D}$ is dense in E. For each $D \in \mathcal{D}$, choose an $x_D \in D$, and let $B = \{x_D \colon D \in \mathcal{D}\}$. Then $B \in \mathcal{B}(X)$; for B is discrete. By our assumption B is countable, and hence so are $\mathcal{D}$ and $\bigcup \mathcal{D}$.

3. COROLLARY. *If X is not ample, then it contains an S-space.*

The following lemma is a special case of Theorem 2.6(i) from [**J**, p. 12].

4. LEMMA. *Let X be not Lindelöf. Then there is a locally countable $Y \subset X$ such that Y is not Lindelöf and $|Y| = \omega_1$.*

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PROOF. Let $\mathcal{U}$ be an open cover of X which has no countable subcover. Then we can find a collection $\{U_\alpha : \alpha < \omega_1\} \subset \mathcal{U}$ where each $U_\beta - \bigcup_{\alpha<\beta} U_\alpha \neq \emptyset$. For every $\beta < \omega_1$, choose an $x_\beta \in U_\beta - \bigcup_{\alpha<\beta} U_\alpha$. The set $Y = \{x_\beta : \beta < \omega_1\}$ has the desired properties.

5. LEMMA. *Let Y be a hereditarily separable space with $|Y| = 2^\omega$. Then there is an uncountable set $E \subset Y$ which contains no subset $B \in \mathcal{B}(Y)$ with $|B| = 2^\omega$.*

PROOF. Let $\mathcal{H}$ be the family of all sets $B \in \mathcal{B}(Y)$ with $|B| = 2^\omega$. Since Y is hereditarily separable, it contains at most $|Y|^\omega = 2^\omega$ closed subsets. Since Y is a T_1-space, $|\mathcal{H}| = 2^\omega$. Now it is easy to construct a set E with $|E| = 2^\omega$ which contains no element of $\mathcal{H}$ (for the details see [**K**, §40, I, Theorem 2]).

6. THEOREM (CH). *A space X is ample if and only if it contains no S-space.*

PROOF. If X contains an S-space, then by Lemma 4, it also contains a locally countable S-space $Y \subset X$ with $|Y| = 2^\omega$. It follows from Lemma 5 that Y, and consequently X, is not ample. The converse is given by Corollary 3.

The following example shows that Theorem 6 cannot be proved without CH.

7. EXAMPLE. By an unpublished result of Szentmiklóssy there is a model M of ZFC+MA+¬CH in which a regular S-space exists. It follows from Lemma 4 that in M there is a regular S-space Y with $|Y| = \omega_1$. By [**MS**] (the lemma following Theorem 2 in §2), in M there exists a subspace X of real numbers such that each subset of X is Borel, and $|X| = \omega_1$. Choose a bijection $f: Y \to X$, and let τ be the weakest topology in Y which refines the given topology of Y, and for which f is continuous. Then each subset of (Y, τ) is Borel, and hence (Y, τ) is ample. However, since X is second countable it is easy to see that (Y, τ) is still a regular S-space.

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