THE SPACES WHICH CONTAIN AN S-SPACE

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ABSTRACT. Under the continuum hypothesis, we show that a $T_1$-space $X$ contains an S-space if and only if there is an uncountable locally countable set $E \subset X$ containing no Borel subset of $X$.

All spaces in this note are $T_1$. A space $X$ is called locally countable if each point has a countable neighborhood. A set $E \subset X$ is called locally countable if it is such as a subspace. An S-space is a hereditarily separable space which is not Lindelöf. We remark that contrary to the usual definition, in this note an S-space is not required to be regular. If $X$ is a space, we denote by $B(X)$ the family of all Borel subsets of $X$, i.e., the $\sigma$-algebra in $X$ generated by the topology of $X$. If $A$ is a set, we denote by $|A|$ its cardinality.

The Zermelo-Fraenkel set theory including the axiom of choice, the continuum hypothesis, and Martin's axiom will be abbreviated as ZFC, CH, and MA, respectively.

1. DEFINITION. A space $X$ is called ample if each uncountable locally countable set $E \subset X$ contains an uncountable subset $B \in B(X)$.

In this definition the word "ample" is used to indicate a certain richness of the Borel structure of $X$. The ample spaces are important in the topological measure theory; for each diffused, regular, Radon measure in an ample space is $\sigma$-finite (see [GP, 2.4]). Thus it appears useful to characterize the ample spaces in terms of much studied S-spaces.

The following lemma was first proved by R. J. Gardner (see [GP, 2.5]).

2. LEMMA. Let $E \subset X$ be a locally countable set containing no uncountable subset $B \in B(X)$. Then $E$ is hereditarily separable.

PROOF. It suffices to show that $E$ is separable. By Zorn's lemma there is a maximal disjoint family $D$ of nonempty countable subsets of $E$ which are open in $E$. By the maximality of $D$, $\bigcup D$ is dense in $E$. For each $D \in D$, choose an $x_D \in D$, and let $B = \{x_D : D \in D\}$. Then $B \in B(X)$; for $B$ is discrete. By our assumption $B$ is countable, and hence so are $D$ and $\bigcup D$.

3. COROLLARY. If $X$ is not ample, then it contains an S-space.

The following lemma is a special case of Theorem 2.6(i) from [J, p. 12].

4. LEMMA. Let $X$ be not Lindelöf. Then there is a locally countable $Y \subset X$ such that $Y$ is not Lindelöf and $|Y| = \omega_1$. 

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PROOF. Let \( \mathcal{U} \) be an open cover of \( X \) which has no countable subcover. Then we can find a collection \( \{ U_\alpha : \alpha < \omega_1 \} \subset \mathcal{U} \) where each \( U_\beta - \bigcup_{\alpha < \beta} U_\alpha \neq \emptyset \). For every \( \beta < \omega_1 \), choose an \( x_\beta \in U_\beta - \bigcup_{\alpha < \beta} U_\alpha \). The set \( Y = \{ x_\beta : \beta < \omega_1 \} \) has the desired properties.

5. LEMMA. Let \( Y \) be a hereditarily separable space with \( |Y| = 2^\omega \). Then there is an uncountable set \( E \subset Y \) which contains no subset \( B \in B(Y) \) with \( |B| = 2^\omega \).

PROOF. Let \( \mathcal{H} \) be the family of all sets \( B \in B(Y) \) with \( |B| = 2^\omega \). Since \( Y \) is hereditarily separable, it contains at most \( |Y|^\omega = 2^\omega \) closed subsets. Since \( Y \) is a \( T_1 \)-space, \( |\mathcal{H}| = 2^\omega \). Now it is easy to construct a set \( E \) with \( |E| = 2^\omega \) which contains no element of \( \mathcal{H} \) (for the details see [K, §40, I, Theorem 2]).

6. THEOREM (CH). A space \( X \) is ample if and only if it contains no \( S \)-space.

PROOF. If \( X \) contains an \( S \)-space, then by Lemma 4, it also contains a locally countable \( S \)-space \( Y \subset X \) with \( |Y| = 2^\omega \). It follows from Lemma 5 that \( Y \), and consequently \( X \), is not ample. The converse is given by Corollary 3.

The following example shows that Theorem 6 cannot be proved without CH.

7. EXAMPLE. By an unpublished result of Szentmiklosi, there is a model \( M \) of ZFC+MA+¬CH in which a regular \( S \)-space exists. It follows from Lemma 4 that in \( M \) there is a regular \( S \)-space \( Y \) with \( |Y| = \omega_1 \). By [MS] (the lemma following Theorem 2 in §2), in \( M \) there exists a subspace \( X \) of real numbers such that each subset of \( X \) is Borel, and \( |X| = \omega_1 \). Choose a bijection \( f : Y \rightarrow X \), and let \( \tau \) be the weakest topology in \( Y \) which refines the given topology of \( Y \), and for which \( f \) is continuous. Then each subset of \( (Y, \tau) \) is Borel, and hence \( (Y, \tau) \) is ample. However, since \( X \) is second countable it is easy to see that \( (Y, \tau) \) is still a regular \( S \)-space.

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REFERENCES


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