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SHORTER NOTES

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A SIMPLE PROOF OF RADÓ'S THEOREM

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In this note we show that Radó's theorem [3] (sometimes called the Radó-Behnke-Stein-Cartan theorem—see [2] and the literature cited there) is an easy consequence of a basic lemma in the analytic theory of Riemann surfaces, often called Weyl's lemma [1].

Radó's theorem reads: *If $f(z)$ is a complex-valued function continuous on the unit disc D and analytic where it is not zero, then it is analytic in all of D.*

Weyl's lemma states: *If ϕ is a real-valued Lebesgue measurable function on D such that $\int\int_D \phi \cdot \Delta\mu \, dx\, dy = 0$ for every C^∞ function μ with compact support contained in D, then ϕ is almost everywhere equal to a harmonic function on D.*

Here is how to deduce Radó's theorem from Weyl's lemma. Let $f(z) = u(z) + iv(z)$; we shall use Weyl's lemma to show that u and v are harmonic on all of D. Let $D = Z \cup N$ where $Z = \{z \in D\colon u(z) = 0\}$ and N is the open subset of D where $u(z)$ is nonzero and, hence, is harmonic. Let $\{U_i\}$ be a countable locally finite open covering of N by disc-like sets, and let $\{e_i(z)\}$ be an associated C^∞ partition of unity. Finally, let $\mu(z)$ be a C^∞ "test function" as above. We have

$$\begin{aligned}\iint_D u \cdot \Delta\mu \, dxdy &= \iint_Z u \cdot \Delta\mu \, dxdy + \iint_N u\Delta\mu \, dxdy \\ &= \iint_N u\Delta\mu \, dxdy\end{aligned}$$

since the integral over Z, where $u(z) = 0$, is zero. But

$$\iint_N u \cdot \Delta\mu \, dxdy = \sum_i \iint_N u \cdot \Delta(e_i\mu) \, dxdy,$$

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and

$$\iint_N u \cdot \Delta(e_i\mu)\,dxdy = \iint_{U_i} u \cdot \Delta(e_i\mu)\,dxdy$$
$$= \iint_{U_i} \Delta u \cdot e_i\mu\,dxdy = 0.$$

(Note that $e_i\mu$ and all its derivatives are zero on ∂U_i.) Thus u is harmonic on D, and similarly v is harmonic on D also.

To finish the proof of Radó's theorem, we must show that u and v are conjugate harmonic functions on all of D. Since u and v are obviously harmonic conjugates off their common zero set, it is automatic by continuity that they are harmonic conjugates throughout D. Thus $f(z) = u(z) + iv(z)$ is analytic on D. Q.E.D.

References

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3. T. Radó, *Über eine nicht-fortsetzbare Riemannsche Mannigfaltigkeit*, Math. Z. **20** (1924), 1–6.

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