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# CYCLIC RELATIONS AND THE GOLDBERG COEFFICIENTS IN THE CAMPBELL-BAKER-HAUSDORFF FORMULA

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ABSTRACT. Recursively defined coefficients for the commutator-free form of the Campbell-Baker-Hausdorff formula are shown to apply to the commutator version of this formula, and a new relation on these coefficients is derived.

Let $x$ and $y$ be noncommuting indeterminates over the rational numbers. The Campbell-Baker-Hausdorff formula asserts that $e^x e^y = e^z$, where $z$ is a formal series in $x$ and $y$ with rational coefficients, and in fact that $z$ is a Lie element, meaning that it may be written as a formal series in iterated commutators of $x$ and $y$.

(a) An attractive paper of Karl Goldberg published twenty-six years ago presents $z$ as a commutator-free series in $x$ and $y$:

$$z = x + y + \sum_{n=2}^{\infty} \sum_{\substack{w \\ |w|=n}} g_w w,$$

where the inner sum is over all words $w = w_1 w_2 \cdots w_n$ having length $|w| = n$ (each $w_i$ is $x$ or $y$), and $g_w$ is a rational number that depends on the word $w$. Goldberg [1] gave very nice formulas expressing the $g_w$ in terms of a family of recursively computable polynomials, formulas that quite efficiently enable the $g_w$ to be calculated for words $w$ of not too great length, e.g., up to length 10. Goldberg also found several identities among the $g_w$.

(b) A classical and often cited result of Dynkin, see [2], gives a more or less explicit presentation of $z$ in iterated commutators,

$$z = \sum_{m=1}^{\infty} \sum_{p_i, q_i} \frac{(-1)^{m-1}}{m \sum_i (p_i + q_i)} \frac{[w]}{p_1! q_1! \cdots p_m! q_m!},$$

where, if the word $w = w_1 w_2 \cdots w_n$ with each $w_i$ an $x$ or a $y$,

$$[w] = [[\ldots[[w_1, w_2], w_3, \ldots], w_n]]$$

is the same word in Lie multiplication. Here $[w_1, w_2] = w_1 w_2 - w_2 w_1$. The formula is awkward to use, though, since the inner sum is over all $p_i, q_i$ with $p_i \geq 0$, $q_i \geq 0$, $p_i + q_i > 0$, and where $w$ depends on the $p_i, q_i$,

$$w = x^{p_1} y^{q_1} x^{p_2} y^{q_2} \cdots x^{p_m} y^{q_m}.$$

Different choices of $p_i, q_i, m$ may lead to terms in the same word, since (for example) $x^1 y^3 = x^1 y^1 x^0 y^2$.

In his paper Goldberg did not consider the commutator form of $z$, but did express a hope that his commutator-free analysis would be useful in the commutator version.

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It seems to have gone unnoticed that his hope is almost immediately realizable, his nice recursive coefficients $g_w$ indeed applying to the commutator version and giving it a computationally better form:

THEOREM 1.

$$z = x + y + \sum_{n=2}^{\infty} \sum_{\substack{w \\ |w|=n}} \frac{g_w}{n}[w]. \tag{1}$$

PROOF. We use standard facts, see [**2**]. On the rational vector space spanned by words $w$ of a fixed length $n > 0$, define linear transformation $\sigma$ by $\sigma(w) = [w]$. The Specht-Wever theorem, see [**2**], asserts that an element $a$ of this vector space is a Lie element if and only if $\sigma(a) = na$. Let $z_n$ be the homogeneous component of $z$ of degree $n$. Then it is known [**2**] that $z_n$ is a Lie element, whence by the Specht-Wever theorem $\sigma(z_n) = nz_n$. With $z_n = \sum_{w,|w|=n} g_w w$, our formula is immediate.

As an application of Theorem 1, we obtain a further identity involving the Goldberg coefficients, one not observed by Goldberg himself. Notation: On the vector space spanned by length $n > 1$ words, let $c$ be the linear transformation that cyclically shifts each word:

$$c(w_1 w_2 \cdots w_n) = w_n w_1 w_2 \cdots w_{n-1}.$$

LEMMA. $[w] + c[w] + c^2[w] + \cdots + c^{n-1}[w] = 0$.

PROOF. Let $w = \hat{w} w_n$ where $\hat{w} = w_1 w_2 \cdots w_{n-1}$. Then $[w] = [\hat{w}]w_n - w_n[\hat{w}] = [\hat{w}]w_n - c([\hat{w}]w_n)$. Applying $c$ repeatedly, then summing and using $c^n =$ identity, the formula follows.

THEOREM 2. *The Goldberg coefficients $g$, taken over all cyclic shifts of a fixed word $w$ of length at least two, sum to zero.*

PROOF. We have $\sigma(z_n) = nz_n$, i.e.,

$$\sum_{\substack{w \\ |w|=n}} g_w w = \sum_{\substack{w \\ |w|=n}} g_w [w].$$

Applying $c^0, c^1, c^2, \ldots, c^{n-1}$, then summing, we get

$$\sum_{\substack{w \\ |w|=n}} g_w \{w + c(w) + \cdots + c^{n-1}(w)\} = 0,$$

which may be rewritten as

$$\sum_{\substack{w \\ |w|=n}} \{g_w + g_{c(w)} + \cdots + g_{c^{n-1}(w)}\} w = 0.$$

Therefore $g_w + g_{c(w)} + \cdots + g_{c^{n-1}(w)} = 0$, as desired.

REMARKS. (i) It is possible to prove Theorem 2 directly from Goldberg's recursion formulas, by a more elaborate argument involving generating functions. (ii) While terms in (1) may combine, e.g., using $[x, y] = -[y, x]$ or the Jacobi identity, there is less difficulty of this type than in the Dynkin formula. (iii) The identities so far known on the Goldberg coefficients are linear; are there nonlinear identities?

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## REFERENCES

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